ESSAYS

ON

POWERS AND THEIR DIFFERENCES.

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THE FIRST ALGEBRAICAL ESSAY.

THE BINOMIAL THEOREM.

INTRODUCTION.

BY the Binomial Theorem, we obtain a simple and general formula, which represents either the result of a constant multiplication of a Binomial quantity by itself, or some Root of such product, or the reciprocal of either of these. All the cases are concisely expressed by the exponential notation. In the first case the exponent or Index is an affirmative whole number, in the second case the exponent is an affirmative Fraction, and in the last case, in which the Theorem represents a Quotient, the exponent is either a negative Fraction, or a negative Integer. In the first of these, the series is often called a power as opposed to a Root, although "a power" is a general

 term

term now used to represent all the cases I have mentioned. The Binomial Theorem extends to all of them with that generality which could never be attained by going through the arithmetical operations denoted by the Indices or Exponents. Thus if $\frac{m}{r}=\frac{\pm n}{r}$, then, $\sqrt[r]{\overline{p+x}\times\overline{p+x}\times\overline{p+x}\times \&c.. \text{ to } n \text{ terms}}$ or $\overline{p+x}^{\frac{n}{r}}$ and $\sqrt[r]{\frac{1}{\overline{p+x}\times\overline{p+x}\times\overline{p+x}\times \&c. \text{ to } n \text{ terms}}}$ or $\overline{p+x}^{\frac{-n}{r}}$ will be represented by the following General Theorem;

$$\overline{p+x}^{\frac{m}{r}}=p^{\frac{m}{r}}+\frac{m}{r}xp^{\frac{m}{r}-1}+\frac{\frac{m}{r}.(\frac{m}{r}-1)x^{2}p^{\frac{m}{r}-2}}{1\,.\,2}+\&c.$$

a Demonstration of this Theorem, which is the subject of the following Essay, was the result, a few years ago, of pursuing the excellent mathematical course, which is delivered by the Rev. Dr. Magee, in the University of Dublin.

Although Sir Isaac Newton discovered, so early as the year 1669, this Theorem for the extraction of Roots of powers by the method of Infinite Series, yet it does not appear that he had ever discovered a proof of the truth of the Theorem, and notwithstanding the Fluxional Demonstrations which later mathematicians had given, it was long observed, that an algebraical rule might more justly be proved by the principles of algebra. Of the above series particular cases only, have, as yet, been algebraically demonstrated, and the other cases have been usually inferred by Induction. Sir Isaac Newton first considered Roots as powers. In all cases he represented

the Index in a Fractional Form, and where the Index was really a Fraction, he applied with success the same form which already had served when the power had an Integral exponent. That this Induction however is not a legitimate proof, will appear from distinguishing the particular cases which are comprehended under this general mode of notation.

When the Index of a power of a Binomial is an affirmative whole number, that power is produced by repeatedly multiplying the Binomial quantity by itself so often as shall make the number of multiplications to be less by one than are the units in the Index of the power: and it follows from the arrangements of common multiplication, that the power of a Binomial quantity whose Index is the whole number m, as $\overline{1+x}|^{m}$, will have the following Form, viz. $1+mx+Cx^{2}+Dx^{3}+Ex^{4}+\&c.$ $+Zx^{m}$. The Index of such a power, as well as every other quantity in numbers, can be represented after the manner of a Fraction as $m=\frac{rm}{r}$. But, if the Index is really a Fraction, the power cannot have arisen as above from a continued multiplication of the Binomial quantity by itself, since a multiplication can no more be repeated a Fractional than a negative number of times. The analogy, therefore, which is founded on the consideration of $\overline{1+x}|^{\frac{m}{r}}$ under the above notion of powers, will be insufficient to determine even the Form of a

power

power of a Binomial quantity in the case of a Fractional or a negative Index; and until we shall have by some means discovered the law of the series which is equal to $\overline{1+x}^{\frac{m}{r}}$, we shall call it the r^{th} Root of the m^{th} power of $\overline{1+x}$, instead of calling it the $\frac{m}{r}^{th}$ power of $\overline{1+x}$, or a power of $1+x$ whose Index is the Fraction $\frac{m}{r}$. And even considering the quantity $\overline{1+x}^{\frac{m}{r}}$ in this point of view as the r^{th} root of $\overline{1+x}^{m}$, it has been usual to assume an infinite series of the above Form, $1+\frac{m}{r}x+cx^2+dx^3+\&c.=\overline{1+x}^{\frac{m}{r}}$; but it should first be made to appear why no Fractional Index shall be found in the series, or why in the r^{th} power of the infinite series, the powers of x which are higher than m will break off. In deducing the law of the indices of the powers of x, I shall not attempt to express a finite quantity $\overline{1+x}^{\frac{m}{r}}$ by a series having an infinite number of terms, which attempt must appear to be impossible, nor in truth shall I assume a polynomial expression for any part of the quantity $\overline{1+x}^{\frac{m}{r}}$, until I shall prove that such expression will have arisen from a previous extraction of Roots. As to the Coefficients of x, it will be sufficient to shew from the extraction of Roots that the Coefficient of the second term

term is $\frac{m}{r}$, since the remaining Coefficients can from hence be discovered by algebraical operations more simple than the extraction of Roots. The Coefficient of the second Term, and the Indices of the Terms, will be the subject of the following Chapter; and I shall reserve for the Second Chapter, the Law of the remaining Coefficients.

CHAP. I.

Of the Law of the Indices of the Terms in the Binomial Theorem.

If r is any affirmative whole Number, the following will be a General Rule for extracting the r^{th} Root of $\overline{1+x}|^m$ or of $1+mx+Cx^2+Dx^3+Ex^4+\&c.+Zx^m$.

Subduct the r^{th} power of the first member of the Root (which first member of the Root we know to be an unit) from the given power, divide the remainder by r times the $\overline{r-1}|^{th}$ power of the first member, the first term of the quotient is the second member of the Root.

Subduct from the given power the r^{th} power of the Binomial found, conceived as the first member of the Root, and divide the remainder by r times the $\overline{r-1}|^{th}$ power of the Binomial found, the first term of the quotient is the third member of the Root.

Subduct from the given power the r^{th} power of the Trinomial found, conceiving the Trinome found to be the first member

member of the Root, divide the remainder as before, then conceive the first, second, third, and fourth members found, to be a Quadrinomial, &c.

This Rule follows from the nature of Involution, for, if you subduct the r^{th} power of the first member of the Root from the r^{th} power of a Binomial, the remainder will be r times the $\overline{r-1}^{th}$ power of the first member of the Root multiplied simply by the second member, + plus certain multiples of powers of the first member, having a lower number than $r-1$ for their Index, and having certain powers of the second member for their Cofactors.

Although this method of extracting Roots supposes that from the power of $\overline{1+x}$ whose Index is the Integer m, there are successively taken the powers of Polynomials having the integer r for the Index of the powers, yet to demonstrate the law of the Indices of the terms in the Root, it will not be necessary to suppose the actual Coefficients of integral powers of polynomials to be determined, but merely the relation of indeterminate Coefficients to be known, which will appear from the following Law.

If to a multinomial of n terms, or to $1+bx+cx^2+dx^3+\&c.+sx^{n-1}$, there be added a term containing the next higher dimension of x as tx^n, then the $\overline{n+1}^{th}$ term of the r^{th} power of the new multinomial exceeds the $\overline{n+1}^{th}$ term of the r^{th} power of the multinomial of n terms, by r times the member added, and the n first

first terms in both the powers are the same. For n being the Index of the term added, it will enter no term of the r^{th} power before the $\overline{n+1}^{th}$, and there it is multiplied (from the nature of Involution) by the product of r multiplied by the $\overline{r-1}^{th}$ power of the first member of the Root, or by the product of r into unity.

EXAMPLE.

$$\overline{\overline{1+bx+cx^2+\&c.+sx^{n-1}}+tx^n}^{\,r} =$$

$$\left\{\begin{array}{ll} \overline{1+bx+\&c.+sx^{n-1}}^{\,r} & =1+Bx+Cx^2+\&c.+Sx^{n-1}+Tx^n+\&c. \\ +r\times tx^n\times\overline{1+bx+\&c.+sx^{n-1}}^{\,r-1} = & \qquad\qquad\qquad +rtx^n+\&c. \\ +\&c. & = \qquad\qquad\qquad\qquad +\&c. \end{array}\right.$$

$$=1+Bx+Cx^2+\&c.+Sx^{n-1}+(T+rt)x^n+\&c.$$

PROP. In the r^{th} Root of the m^{th} power of the quantity $\overline{1+x}$, that is in $\overline{1+x}^{\frac{m}{r}}$, there will be found no Fractional powers of the second member of the Binomial quantity, and the Indices of the powers of x in the successive terms will be the series of natural numbers.

Where

Where the Index of the power of a Binomial is a positive integer, this Law will appear from the arrangements of common multiplication: That the Law is the same where the Index is an affirmative or negative Fraction, will follow from the rule which I have given for extracting the r^{th} Root of the m^{th} power of a Binomial, viz. by shewing that if from $\overline{1+x}^{\,m}$ or from $1+mx+Cx^2+Dx^3+Ex^4+\&c.+Zx^m$ you subduct 1^r, in the remainder, the simple dimension of x is the lowest, and the second member of the Root is $\frac{m}{r}x$. That by subducting $\overline{1+\frac{m}{r}x}^{\,r}$ from the given power, in the remainder the second dimension of x is the lowest and the next member of the Root is of two dimensions. That by subducting $\overline{(1+\frac{m}{r}x)+cx^2}^{\,r}$, in the remainder all the dimensions before the third will be destroyed, and the new term is of three dimensions. That by subducting $\overline{(1+\frac{m}{r}x+cx^2)+dx^3}^{\,r}$ from the given power, in the remainder all the dimensions below the fourth will be destroyed and so on, ad libitum. This will now be shewn by demonstrating that such Coefficients in the Subtrahends are equal to the corresponding Coefficients in the given power, since r times the member added to the polynomial, is their common excess above equals.

Demonstration. The first member of the power being Unit, the Subduction of 1^{r} and the division of the first term of the remainder by r gives $\frac{m}{r}x$ for the second member of the Root; by subducting again the r^{th} power of the Binomial quantity $1+\frac{m}{r}x$ from the given power, the two first terms of the power are destroyed, for by the above method of extraction of Roots, the second member of the Binomial was derived by dividing the second member of the series by r, and this is again multiplied by r in the second member of the r^{th} power of the Binomial found, therefore the first and second terms of the r^{th} power of the Binomial found being subducted respectively from the first and second terms of the given series, the first dimension of x will not appear in the remainder, and therefore the next term is of two dimensions. The two first terms of the r^{th} power of the Trinomial are the same as those of that of the Binomial found. But from the above nature of Involution, the third term of the r^{th} power of the Trinomial is greater than the third term of the r^{th} power of the Binomial by an excess which is the third term of the Root multiplied by r, and from the above nature of extraction of Roots, the third term of the given series is also greater than the third term of the r^{th} power of the Binomial found, by an excess which is also the third term of the Root

Root multiplied by r. Therefore the third term of the r^{th} power of the Trinomial and the third term of the given series have the same excess above the third term of the r^{th} power of the Binomial, and therefore are equal. Therefore after the subduction, the three first terms will be exterminated, and the next member is of three dimensions.

EXAMPLES.†

$$\text{Given power is } \overline{1+x}^{m} = 1 + mx + Cx^2 + Dx^3 + \&c.$$
$$\text{Subduct } \quad 1$$
$$* \quad mx + \&c.$$
$$r)mx(\tfrac{m}{r}x$$

$$\text{Given power is } 1 + mx + Cx^2 + Dx^3 + \&c.$$
$$\text{Subduct } \overline{1+\tfrac{m}{r}x}^{r} = 1 + mx + \mathrm{c}x^2 + \&c.$$
$$* \quad * \quad rcx^2 + \&c.$$
$$r)rcx^2(cx^2$$

$$\text{Given power is } 1 + mx + \quad Cx^2 \quad + \quad Dx^3 + \&c.$$
$$\text{Subduct } \overline{(1+\tfrac{m}{r}x) + cx^2}^{r} = 1 + mx + (\mathrm{c}+rc)x^2 + \mathrm{d}x^3 + \&c.$$
$$* \quad * \quad * \quad rdx^3 + \&c.$$
$$r)rdx^3(dx^3$$

 In

† N. B. In this and in the subsequent Example, the small Roman letters, c, d, &c. t, v, &c. which are used in the Subtrahends, are to be distinguished from the small Italics, *c*, *d*, &c. *t*, *v*, &c. which are substituted in the Remainders.

In general if the n first terms of the r^{th} power of a multinomial of n terms derived in this way be equal to the n first terms of the given series (as has been proved to the Case of the Trinomial), and the excess of the $\overline{n+1}^{th}$ term of the given series above the $\overline{n+1}^{th}$ term of the said r^{th} power of the multinomial of n terms, be made according to the above method of Extraction to be equal to r times the new member of the Root, (this new member to be added to the former multinomial). Then, in subducting from the given power the r^{th} power of the new multinomial, the n first terms are the same as before, and the $\overline{n+1}^{th}$ term of the r^{th} power of the new multinomial will be equal to the $\overline{n+1}^{th}$ term of the series. For both of those terms exceed the $\overline{n+1}^{th}$ term of the r^{th} power of the former multinomial by r times the term added. Hence, in the remainder the $n+1$ first terms will be destroyed, and the Index of x will be higher by an Unit in the new member of the Root than in the preceding member, and therefore vx^{n+1} will be the $n+2$ term if tx^{n} be the $n+1$ term of the Root (as we have supposed) which will appear by the following General Example, viz.

GENERAL

GENERAL EXAMPLES

$$\text{Given power } \overline{1+x}^{m} = 1+mx+Cx^{2}+\&c.+Sx^{n-1}+Tx^{n}+\&c.$$

$$\overline{1+\tfrac{m}{r}x+cx^{2}+\&c.+sx^{n-1}}^{\,r} = 1+mx+Cx^{2}+\&c.+Sx^{n-1}+tx^{n}+\&c.$$

$$\text{Remainder} = * \quad * \quad * \quad * \quad *(T-t)x^{n}+\&c.$$

$$(T-t)x^{n} = rtx^{n}$$

$$\text{Given power } \overline{1+x}^{m} = 1+mx+\&c.+Sx^{n-1}+Tx^{n}+Vx^{n+1}+\&c.$$

$$\overline{1+\tfrac{m}{r}x+\&c.+sx^{n-1}+tx^{n}}^{\,r} = 1+mx+\&c.+Sx^{n-1}+(t+rt)x^{n}+vx^{n+1}+\&c.$$

$$\text{Remainder} = * \quad * \quad * \quad * \ (T-\overline{t+rt})x^{n}+(V-v)x^{n+1}+\&c.$$

$$(V-v)x^{n+1} = rvx^{n+1}$$

In the former of these Remainders $T-t=rt$, and hence in the latter Remainder $T=t+rt \therefore T-\overline{t+rt}=0$, Therefore the Coefficient of every power of x whose Index is not greater than n will be destroyed, and the lowest Index in the latter Remainder is $n+1$, therefore the Law $x^{0}, x^{1}, x^{2}, x^{3}$, &c. x^{n}, x^{n+1}, &c. will be the General Law of the several terms of the Root.

To

To elucidate this Example, by applying it to the Extraction of Roots of numbers, we shall extract the square root of 1.331, or of the cube of 1.1, where $x=\frac{1}{10}$ and 1.331 is arranged according to the Indices of powers in the Decimal series, and may thus be otherwise expressed $1+\frac{3}{10}+\frac{3}{100}+\frac{1}{1000}$. Now m being $=3$, and $r=2$, we shall apply the foregoing Example as follows :

$$\text{Given Power } \left(1+\tfrac{1}{10}\right)^{3} = 1+\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{1}{1000}$$

$$1^{2} = 1$$

$$\text{Remainder} \quad * \quad \tfrac{3}{10}+\&c.$$

$$2)\tfrac{3}{10}\left(\tfrac{3}{2}\times\tfrac{1}{10}\right.$$

$$\text{Given Power } \overline{1+\tfrac{1}{10}}^{\,3} = 1+3\times\tfrac{1}{10}+3\times\tfrac{1}{100}+\tfrac{1}{1000}$$

$$\overline{1+\tfrac{3}{2}\times\tfrac{1}{10}}^{\,2} = 1+3\times\tfrac{1}{10}+\tfrac{9}{4}\times\tfrac{1}{100}$$

$$\text{Remainder} \quad * \quad * \quad \tfrac{3}{4}\times\tfrac{1}{100}+\&c.$$

$$2)\tfrac{3}{4}\times\tfrac{1}{100}\left(\tfrac{3}{8}\times\tfrac{1}{100}\right.$$

$$\text{Given Power } \overline{1+\tfrac{1}{10}}^{\,3} = 1+3\times\tfrac{1}{10}+3\times\tfrac{1}{100}+\tfrac{1}{1000}$$

$$\overline{1+\tfrac{3}{2}\times\tfrac{1}{10}+\tfrac{3}{8}\times\tfrac{1}{100}}^{\,2} = 1+3\times\tfrac{1}{10}+3\times\tfrac{1}{100}+\tfrac{9}{8}\times\tfrac{1}{1000}+\&c.$$

$$\text{Remainder} \quad * \quad * \quad * \quad -\tfrac{1}{8}\times\tfrac{1}{1000}-\&c.$$

$$2)-\tfrac{1}{8}\times\tfrac{1}{1000}\left(-\tfrac{1}{16}\times\tfrac{1}{1000}\right.$$

Hence $\overline{1+\frac{1}{10}}^{\,\frac{3}{2}} = 1+\frac{3}{2}\times\frac{1}{10}+\frac{3}{8}\times\frac{1}{100}-\frac{1}{16}\times\frac{1}{1000}$, &c. and putting for $\frac{3}{2}\times\frac{1}{10}$, $\frac{3}{8}\times\frac{1}{100}$, $-\frac{1}{16}\times\frac{1}{1000}$, &c. the Decimals, 0.15, 0.00375, —0.0000625, &c. we have $\overline{1+\frac{1}{10}}^{\,\frac{3}{2}}=1.1536$&c. and this is the result which

which would be derived by extracting the Square Root of 1.331 by the known method which depends upon trial.

In like manner, we extract the Cube Root of $\overline{1+\frac{1}{10}}^{\,4}$ or of 1.4641, which may be otherwise expressed $1+\frac{4}{10}+\frac{6}{100}+\frac{4}{1000}+\frac{1}{10000}$, viz.

$$\begin{array}{ll}
\text{Given Power is} & 1+\frac{4}{10}+\frac{6}{100}+\frac{4}{1000}+\frac{1}{10000} \\
1^{3} & =1 \\
\hline
\text{Remainder} = & * \quad 4\times\frac{1}{10}+\&c. \\
 & \quad 3\,)\,4\times\frac{1}{10}\,(\,\frac{4}{3}\times\frac{1}{10}
\end{array}$$

$$\begin{array}{ll}
\text{Given Power is} & 1+4\times\frac{1}{10}+6\times\frac{1}{100}+4\times\frac{1}{1000}+\frac{1}{10000} \\
\overline{1+\frac{4}{3}\times\frac{1}{10}}^{\,3} = & 1+4\times\frac{1}{10}+\frac{16}{3}\times\frac{1}{100}+\frac{64}{27}\times\frac{1}{1000} \\
\hline
\text{Remainder} = & * \quad * \quad \frac{2}{3}\times\frac{1}{100}+\&c. \\
 & \quad\quad 3)\,\frac{2}{3}\times\frac{1}{100}\,(\,\frac{2}{9}\times\frac{1}{100}
\end{array}$$

$$\begin{array}{ll}
\text{Given Power is} & 1+4\times\frac{1}{10}+6\times\frac{1}{100}+4\times\frac{1}{1000}+\frac{1}{10000} \\
\overline{1+\frac{4}{3}\times\frac{1}{10}+\frac{2}{9}\times\frac{1}{100}}^{\,3} = & 1+4\times\frac{1}{10}+6\times\frac{1}{100}+\frac{112}{27}\times\frac{1}{1000}+\&c. \\
\hline
\text{Remainder} = & * \quad * \quad * \quad -\frac{4}{27}\times\frac{1}{1000}\ \&c. \\
 & \quad\quad\quad 3)\,-\frac{4}{27}\times\frac{1}{1000}\,(\,-\frac{4}{81}\times\frac{1}{1000}
\end{array}$$

Therefore $\overline{1+\frac{1}{10}}^{\,\frac{4}{3}} = 1+\frac{4}{3}\times\frac{1}{10}+\frac{2}{9}\times\frac{1}{100}-\frac{4}{81}\times\frac{1}{1000}+$&c. and substituting for the Fractions the equivalent Decimals, we shall have the above expression $=1.135$&c. the same which would result from the usual mode of extracting the Cube Root of a number by trials.

I have applied the general Example to the extraction of the Square and Cube Roots, merely to elucidate the method by which

which I have deduced the Law of the Indices in the algebraical operations, and not for the purpose of giving a general rule for numerical extraction of Roots. For, I have supposed certain arrangements in $\overline{1+x}^{m}$ which will not always apply to $\overline{1+\frac{1}{10}}^{m}$, since the dispositions by decimal places in numbers, and those of the letters in algebra are usually different. In the latter case the characters or letters are indeterminate, and therefore no powers but those which are homologous are capable of coalescing under the same coefficient. In the Involution and Evolution of numbers, the members are multiples of terms in the decimal series of powers, the Indices of which powers are indeed in arithmetical progression like the Indices of x in the algebraical formula, but the multiples are denoted by the digits, although, if the algebraical arrangements were followed, they would be denoted by numbers or coefficients, which may themselves contain powers of 10. Hence the powers of the second member of the Root, and the coefficients of those powers, which are kept distinct in the algebraical arrangement, will coalesce into one number in the numerical notation, and the necessary preservation of places and distances will prevent the members of a power of a Binomial number from being arranged according to the Indices of the second member of the Root. Thus $\overline{1+\frac{1}{10}}^{5}$ according to the natural places, or the numerical arrangements of the powers in the decimal series, is 1.61051, or $1+\frac{6}{10}+\frac{1}{100}+\frac{0}{1000}+\frac{5}{10000}+\frac{1}{100000}$.

But

But according to the algebraical disposition of the terms, $\overline{1+\frac{1}{10}}^{5}$ would be thus expressed $1+5\times\frac{1}{10}+10\times\frac{1}{100}+10\times\frac{1}{1000}+5\times\frac{1}{10000}+\frac{1}{100000}$.

The general Example will apply to the latter disposition, but if we use the former arrangement the Extraction of the Square Root will depend upon trial.

COR. I. We have hitherto considered the Indices of x in the r^{th} Root of $\overline{1+x}^{m}$, or in $\overline{1+x}^{\frac{m}{r}}$; but if the Binomial is $p+x$, by resolving it into $p\times\left(1+\frac{x}{p}\right)$, the same proof will extend to the Law of the Indices of the second member; and therefore $\left(1+\frac{x}{p}\right)^{\frac{m}{r}}=1+\frac{mx}{rp}+c\frac{x^2}{p^2}+\&c.$ Hence we shall have $\overline{p+x}^{\frac{m}{r}}=p^{\frac{m}{r}}\times\left(1+\frac{x}{p}\right)^{\frac{m}{r}}$ $=p^{\frac{m}{r}}+\frac{m}{r}x\frac{p^{\frac{m}{r}}}{p}+cx^2\frac{p^{\frac{m}{r}}}{p^2}+\&c.+tx^n\frac{p^{\frac{m}{r}}}{p^n}+\&c.=p^{\frac{m}{r}}+\frac{m}{r}xp^{\frac{m}{r}-1}+cx^2p^{\frac{m}{r}-2}+\&c.+tx^np^{\frac{m}{r}-n}+\&c.$

In the last step of this proof I have supposed the equality of $\frac{p^{\frac{m}{r}}}{p^n}$ and $p^{\frac{m}{r}-n}$, which perhaps may require a Demonstration when $\frac{m}{r}$ is a Fraction. If the Indices of p in the Dividend and Divisor were Integers, it would follow from notation that the subduction of the Index of the Divisor from that of the Dividend should denote a Division; viz. if m and nr be Integers, we shall have $\frac{p^m}{p^{nr}}=p^{m-nr}$; and hence we shall prove that the subduction of the Index of p in the Divisor from the Index of p in the Dividend will effect a Division, although p should

have a Fractional Index, for since $\frac{p^m}{p^{nr}} = p^{m-nr} \therefore \sqrt[r]{\frac{p^m}{p^{nr}}} = \sqrt[r]{p^{m-nr}}$ that is $\frac{p^{\frac{m}{r}}}{p^n} = p^{\frac{m-nr}{r}}$ = or $= p^{\frac{m}{r}-n}$ Thus although we cannot consider the Dividend $p^{\frac{m}{r}}$, whose Index is the Fraction $\frac{m}{r}$, to have been produced by a number of multiplications as in the case where the Index is an Integer, yet the same Law of Division through the subduction of the Indices is shewn to apply in the one case as well as in the other, by considering the Dividend and Divisor under the form of the r^{th} Roots of powers of p whose exponents are Integers; we effect this by the known method of reducing the Radicals to the same Denomination which is the necessary step for their Multiplication or Division.

Cor. II. From the Cases where $\frac{m}{r}$ is affirmative, we shall by algebraical Division be able to prove that the Law of the Terms will be the same when the Index of the power is negative. In this case we have the negative power $\overline{p+x}^{\,-\frac{m}{r}} =$

$$\frac{1}{(p+x)^{\frac{m}{r}}} = \frac{1}{p^{\frac{m}{r}} + \frac{m}{r}xp^{\frac{m}{r}-1} + cx^2p^{\frac{m}{r}-2} + \&c.} = 1 - \frac{\frac{m}{r}x}{p^{\frac{m}{r}+1}} + \left(\frac{m^2}{r^2} - c\right)\frac{x^2}{p^{\frac{m}{r}+2}} + \&c. =$$

$$p^{-\frac{m}{r}} - \frac{m}{r}xp^{-\frac{m}{r}-1} + \left(\frac{m^2}{r^2} - c\right)x^2p^{-\frac{m}{r}-2} + \&c.$$

Cor.

Cor. III. We have now deduced the Law of the Indices of the terms from the extraction of Roots, and from hence also the Coefficients of the second and third terms where the Index of the power is a Fraction will be known from the cases of Involution, wherein the Index of the power being an affirmative Integer, the dependence of the Coefficient on the Index is known.

Thus $rb = m$ therefore $b = \frac{m}{r}$ and in negative powers $0 - \frac{m}{r} = -\frac{m}{r}$

Also $rc = C - c = \frac{m^2 - m}{1.2} - \frac{r^2 - r}{1.2} b^2$ therefore $c = \frac{m^2 - mr}{1.2.r^2}$

and in negative powers $\frac{m^2}{r^2} - c = \frac{2m^2 - m^2 + mr}{1.2.r^2} = \frac{m^2 + mr}{1.2.r^2}$

Having now proved that $\overline{p+x}^{\pm\frac{m}{r}}$ will be of the same form, whether the Index is affirmative or negative, I shall hereafter, for the sake of convenience, put $\frac{m}{r}$ instead of $\pm\frac{m}{r}$, and then the general Index $\frac{m}{r}$ will denote any positive or negative Fraction or whole Number.

If $\frac{m}{r}$ is a surd, we can find rational numbers which will approach it as near we please, and although we cannot conclude from the rules which I have given that $\overline{p+x}^{\frac{m}{r}}$ will be of the above form, when $\frac{m}{r}$ is a surd, since no such arithmetical process as Involution, Evolution, or Division, can in that case be understood, yet the above forms will apply to the powers which have the approximate value of $\frac{m}{r}$ for an Index.

CHAP. II.

Of the Co-efficients of the Terms in the Binomial Theorem.

From the following proposition two corollaries are easily deduced, which La Croix has proved by expanding a complex differential formula, which he has made to depend on the co-efficients of the terms in the Binomial Theorem. (See his Diff. Meth. vol. 3. pag. 7.) But as I shall shew an immediate transition to the unciæ of powers from those corollaries, I shall simply demonstrate them by means of the following independent proposition.

I shall first observe that the terms of an arithmetical series are usually represented as Binomials, whose first members are the constant basis of the progression, and whose second members are the variable multiples of the common difference: but we shall avoid any complex substitutions, if we make the next lesser term of the arithmetical series the first member, and the common difference to be the second member of the Binomial.

Prop. If o, p, q. r, s, &c. are the terms of an arithmetical series whose common difference is d, then will $p+d$, $q+d$, $r+d$, $s+d$, $t+d$, &c. be respectively equal to the corresponding terms of the series, o, p, q, r, s, &c. If we take the differences of the $\frac{m}{r}^{th}$ powers of the Binomials, and the differences of these differences, or the second differences, &c. the

the first members of the n^{th} differences of the $\frac{m}{r}^{th}$ powers of the Binomials, are the $(\frac{m}{r}-n)^{th}$ powers of terms in arithmetical progression, with a constant co-efficient, viz. $\frac{m}{r}d.$ $(\frac{m}{r}-1)d.$ $(\frac{m}{r}-2)d.$ &c. $(\frac{m}{r}-\overline{n-1})d.$ viz.

THE POWERS OF NUMBERS IN ARITHMETICAL PROGRESSION.

$$o^{\frac{m}{r}}=\overline{p+d}^{\frac{m}{r}}=p^{\frac{m}{r}}+\tfrac{m}{r}dp^{\frac{m}{r}-1}+Cd^2p^{\frac{m}{r}-2}+\&c.$$
$$p^{\frac{m}{r}}=\overline{q+d}^{\frac{m}{r}}=q^{\frac{m}{r}}+\tfrac{m}{r}dq^{\frac{m}{r}-1}+Cd^2q^{\frac{m}{r}-2}+\&c.$$
$$q^{\frac{m}{r}}=\overline{r+d}^{\frac{m}{r}}=r^{\frac{m}{r}}+\tfrac{m}{r}dr^{\frac{m}{r}-1}+Cd^2r^{\frac{m}{r}-2}+\&c.$$
$$r^{\frac{m}{r}}=\overline{s+d}^{\frac{m}{r}}=s^{\frac{m}{r}}+\tfrac{m}{r}ds^{\frac{m}{r}-1}+Cd^2s^{\frac{m}{r}-2}+\&c.$$
$$s^{\frac{m}{r}}=\overline{t+d}^{\frac{m}{r}}=t^{\frac{m}{r}}+\tfrac{m}{r}dt^{\frac{m}{r}-1}+Cd^2t^{\frac{m}{r}-2}+\&c.$$
$$t^{\frac{m}{r}}=\&c.$$

FIRST DIFFERENCES.	SECOND DIFFERENCES.
$\frac{m}{r}dp^{\frac{m}{r}-1}+Cd^2p^{\frac{m}{r}-2}+\&c.$	
	$\frac{m}{r}d.\ (\frac{m}{r}-1)\ d\ q^{\frac{m}{r}-2}+\&c.$
$\frac{m}{r}dq^{\frac{m}{r}-1}+Cd^2q^{\frac{m}{r}-2}+\&c.$	
	$\frac{m}{r}d.\ (\frac{m}{r}-1)\ d\ r^{\frac{m}{r}-2}+\&c.$
$\frac{m}{r}dr^{\frac{m}{r}-1}+Cd^2r^{\frac{m}{r}-2}+\&c.$	
	$\frac{m}{r}d.\ (\frac{m}{r}-1)\ d\ s^{\frac{m}{r}-2}+\&c.$
$\frac{m}{r}ds^{\frac{m}{r}-1}+Cd^2s^{\frac{m}{r}-2}+\&c.$	
	$\frac{m}{r}d.\ (\frac{m}{r}-1)\ d\ t^{\frac{m}{r}-2}+\&c.$
$\frac{m}{r}dt^{\frac{m}{r}-1}+Cd^2t^{\frac{m}{r}-2}+\&c.$	

DEMONSTRATION.

Demonstration.—The first differences of the $\frac{m}{r}^{th}$ powers of the Binomials are derived by taking away the first members of those powers, for this is to subduct the $\frac{m}{r}^{th}$ powers of the next lesser Binomials. The second differences of the $\frac{m}{r}^{th}$ powers of the Binomials (or the differences of the first differences) will be equal to the sums of the separate differences of the separate members of the first differences, which members being the powers of terms in an arithmetical series, the indices of which powers are $\frac{m}{r}—1$, $\frac{m}{r}—2$, $\frac{m}{r}—3$, &c. the first differences of these will be known from the first differences of the $\frac{m}{r}^{th}$ powers: For, if $p^{\frac{m}{r}}—q^{\frac{m}{r}}=\frac{m}{r}dq^{\frac{m}{r}-1}+Cd^2q^{\frac{m}{r}-2}+$&c. then will $p^{\frac{m}{r}-1}—q^{\frac{m}{r}-1}=(\frac{m}{r}—1)dq^{\frac{m}{r}-2}+\underset{2}{C}d^2q^{\frac{m}{r}-3}+$&c.; and $p^{\frac{m}{r}-2}—q^{\frac{m}{r}-2}=(\frac{m}{r}—2)dq^{\frac{m}{r}-3}+\underset{3}{C}d^2q^{\frac{m}{r}-4}+$&c.

But there are wanting none except the highest members of the expressions for the first differences, in order to obtain the highest members of the expressions for the second differences, and none except the highest members of the n-1^{th} differences in order to obtain the highest members of the n^{th} differences, for the differences of the highest members contain higher powers than the differences of the lower members.

Since then, the highest members of the first differences of the $\frac{m}{r}^{th}$ powers are $\frac{m}{r}d$ multiplied by the powers of numbers

bers in arithmetical progression, whose Index is $\frac{m}{r}-1$, and since the highest members of those $\frac{m}{r}-1^{th}$ powers are $(\frac{m}{r}-1)\,d$ multiplied by powers whose index is $\frac{m}{r}-2$. Therefore the highest members of the second differences will be the common multiplier $\frac{m}{r}d.\ (\frac{m}{r}-1)d$ multiplied by powers whose Index is $\frac{m}{r}-2$. Again the highest members of the differences of the $\frac{m}{r}-2$ powers of numbers in arithmetical progression are $(\frac{m}{r}-2)d.$ multiplied by the powers whose Index is $(\frac{m}{r}-3)$. Therefore the highest members of the third differences of the $\frac{m}{r}^{th}$ powers $=\frac{m}{r}d(\frac{m}{r}-1)d.(\frac{m}{r}-2)d.$ multiplied by the $\frac{m}{r}-3$ powers of numbers in arithmetical series.

GENERAL EXAMPLE.

The $n-1^{th}$ differences.

$$\frac{m}{r}d\,.\,(\tfrac{m}{r}-1)d\,..\,\&c.\ (\tfrac{m}{r}-\overline{n-2})\,ds^{\frac{m}{r}-\overline{n-1}}+\&c.$$
$$\frac{m}{r}d\,.\,(\tfrac{m}{r}-1)d\,..\,\&c.\ (\tfrac{m}{r}-\overline{n-2})\,dt^{\frac{m}{r}-\overline{n-1}}+\&c.$$
&c.

The n^{th} differences or the differences of the $n-1^{th}$ differences.

$$\frac{m}{r}d.(\tfrac{m}{r}-1)\,d\,..\,\&c.\ (\tfrac{m}{r}-\overline{n-2})\,d.\ (\tfrac{m}{r}-\overline{n-1})\,dt^{\frac{m}{r}-n}+\&c.$$
&c.

In

In general, if the highest members of the $\overline{n-1}^{th}$ differences of the $\frac{m}{r}^{th}$ powers of the terms of the arithmetical series of Binomials be (as has been shewn in the first, second, and third differences) $=\frac{m}{r}d.\ (\frac{m}{r}-1)d.$ &c. $(\frac{m}{r}-n-2)d$ multiplied by the powers of numbers in arithmetical progression whose index is $\frac{m}{r}-\overline{n-1}$, then, the highest members of the n^{th} differences will be equal to the common co-efficient multiplied by the highest members of the differences of $\frac{m}{r}-\overline{n-1}^{th}$ powers of terms in arithmetical series, $=\frac{m}{r}d.\ (\frac{m}{r}-1)d.\ (\frac{m}{r}-2)d.$ &c. $(\frac{m}{r}-\overline{n-2})d.\ (\frac{m}{r}-\overline{n-1})d.$ multiplied by the $\frac{m}{r}-n^{th}$ powers of numbers in arithmetical progression.

Thus in the Examples the quantities

$\frac{m}{r}dt^{\frac{m}{r}-1}$,

$\frac{m}{r}d.(\frac{m}{r}-1)\,dt^{\frac{m}{r}-2}$,

$\frac{m}{r}d.(\frac{m}{r}-1)d.(\frac{m}{r}-2)dt^{\frac{m}{r}-3}$,

$\frac{m}{r}d.(\frac{m}{r}-1)d..\&c.(\frac{m}{r}-\overline{n-1})dt^{\frac{m}{r}-n}$ are found in the series of the first members of the first, second, third, and n^{th} orders of differences of the $\frac{m}{r}^{th}$ powers of the terms in the arithmetical series, in which t is a term, and d the common difference.

COR.

Cor. I. If instead of o, p, q, r, &c. the terms of the arithmetical series be supposed $x+o$, $x+p$, $x+q$, $x+r$, &c. then the first members of the n^{th} differences of the $\frac{m}{r}^{th}$ powers will be $\frac{m}{r}d\ (\frac{m}{r}-1)d.\ (\frac{m}{r}-2)d.$ &c. $(\frac{m}{r}-\overline{n-1})d$ multiplied by the $\frac{m}{r}-n^{th}$ powers of Binomials in the arithmetical progression $x+o$, $x+p$, $x+q$, &c, and if the terms in each difference are made to proceed according to the powers of x, the highest member of each of the n^{th} differences will be constant, and equal to $\frac{m}{r}d.\ (\frac{m}{r}-1)d.\ (\frac{m}{r}-2)d$ &c. $(\frac{m}{r}-\overline{n-1})\ dx^{\frac{m}{r}-n}$.

Cor. II. The second differences of the Squares, the third differences of the Cubes, and in general the n^{th} differences of the n^{th} powers of numbers in arithmetical progression will be constant and equal to the product of the digits from the Index to unity, multiplied by the n^{th} power of the common difference of the arithmetical series, or $=1.2.3.$ &c. $n \times d^{n}$. For, each order of differences being made up of the separate differences of the separate members of the preceding order, and the indices of the members diminished by unit being the highest indices in the differences of those members, when those indices are equal to cypher, the members will be constant in which cypher is the Index of the terms. In each difference

the constant members of the preceding order of differences are destroyed, therefore in the n^{th} differences the first terms alone will remain, and the indices being $=n-n$ the powers will be equal to unit. Therefore the n^{th} differences of the n^{th} powers, are reduced to the constant quantity $nd.(n-1)d.(n-2)d.\&c..3.d.2d.1d=1.2.3.\&c..(n-2).(n-1).n \times d^n$.

Sir Isaac Newton's Binomial Theorem may be deduced from a simple algebraical Equation, derived from those Corollaries and from what has been shewn of the form of $\overline{x+q}|^{\frac{m}{r}}$, by substituting merely the terms of an arithmetical series for q, viz.

$$\overline{x+q}|^{\frac{m}{r}}=x^{\frac{m}{r}}+bqx^{\frac{m}{r}-1}+cq^2x^{\frac{m}{r}-2}+dq^3x^{\frac{m}{r}-3}+\&c.+tq^nx^{\frac{m}{r}-n}+\&c.$$

$$\overline{x+4}|^{\frac{m}{r}}=x^{\frac{m}{r}}+b4.x^{\frac{m}{r}-1}+c.4^2x^{\frac{m}{r}-2}+d.4^3x^{\frac{m}{r}-3}+\&c.+t.4^n.x^{\frac{m}{r}-n}+\&c.$$

$$\overline{x+3}|^{\frac{m}{r}}=x^{\frac{m}{r}}+b.3x^{\frac{m}{r}-1}+c.3^2.x^{\frac{m}{r}-2}+d3^3.x^{\frac{m}{r}-3}+\&c.+t.3^n.x^{\frac{m}{r}-n}+\&c.$$

$$\overline{x+2}|^{\frac{m}{r}}=x^{\frac{m}{r}}+b.2x^{\frac{m}{r}-1}+c.2^2.x^{\frac{m}{r}-2}+d.2^3.x^{\frac{m}{r}-3}+\&c.+t.2^n.x^{\frac{m}{r}-n}+\&c.$$

$$\overline{x+1}|^{\frac{m}{r}}=x^{\frac{m}{r}}+b.1.x^{\frac{m}{r}-1}+c.1^2.x^{\frac{m}{r}-2}+d.1^3.x^{\frac{m}{r}-3}+\&c.+t.1^n.x^{\frac{m}{r}-n}+\&c.$$

Here the n^{th} powers of the natural numbers multiplied by t will be the co-efficients of $x^{\frac{m}{r}-n}$, and since the terms which contained

contained the first, second, third, &c. to the $n-1^{th}$ powers of the natural numbers will not be found in the n^{th} differences of those series, (the n^{th} differences of powers whose Index is less than n, being differences of common differences,) therefore the first members in all the n^{th} differences of the $\frac{m}{r}^{th}$ powers of $x+4$, $x+3$, $x+2$, &c. will be $= t$ multiplied by the n^{th} differences of the n^{th} powers of natural numbers $\times x^{\frac{m}{r}-n}$ or (by Cor. 2.) $= t \times 1.2.3.\ \&c.\ n \times x^{\frac{m}{r}-n}$. But (by Cor. 1.) those common first members $= \frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).\&c.(\frac{m}{r}-\overline{n-1})x^{\frac{m}{r}-n}$. therefore $t \times 1.2.3.\&c.n \times x^{\frac{m}{r}-n} = \frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).\&c.(\frac{m}{r}-\overline{n-1}) \times x^{\frac{m}{r}-n} \therefore t = \dfrac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)..\&c.(\frac{m}{r}-\overline{n-1}).}{1.\ 2.\ 3..\ \&c.\ n}$

Therefore $\overline{x+q}^{\frac{m}{r}} = x^{\frac{m}{r}} + \frac{m}{r}qx^{\frac{m}{r}-1} + \dfrac{\frac{m}{r}.(\frac{m}{r}-1)}{1.2}q^2x^{\frac{m}{r}-2} + \&c. + \dfrac{\frac{m}{r}.(\frac{m}{r}-1).\&c.(\frac{m}{r}-\overline{n-1})}{1.\ 2..\ \&c.\ n}q^n x^{\frac{m}{r}-n} + \&c.$ Q. E. D.

NOTE. In proving the Binomial Theorem above, I have equated the two expressions for the common first members of all the n^{th} differences of the $\frac{m}{r}^{th}$ powers. For, the differences themselves are indentical, being changed only in form without

altering the value of x, and the highest powers of x being the same in both those expressions, if we divide by those powers, then the common first members are the only members in both which do not contain negative powers of x, and therefore the common first members in both expressions are identical.

Newton, in his letter of the 13th of June, 1676, to Mr. Oldenburg, the Secretary to the Royal Society of London, has expressed the Binomial Theorem in the following form, viz. $\overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$ where $P+PQ$ signifies a quantity of which some Root, or some dimension, or some Root of a dimension is to be investigated, also P denotes the first term of that quantity, Q the remaining terms divided by the first, and $\frac{m}{n}$ the numeral Index of the dimension of $P+PQ$, whether that dimension be a whole or a broken quantity, affirmative or negative, and A, B, C, D, &c. are used for the terms found in the progress of the operation, that is A for the first term $P^{\frac{m}{n}}$, and B for the second term $\frac{m}{n}AQ$, &c. See Commercuim Epistolicum, No. XLVIII.

In the 45th Proposition of the First Book of the Principia, Newton gives the same Theorem in the following Terms, viz. $\overline{T-X}^{n} = T^{n} - nXT^{n-1} + \frac{n^2-n}{2}X^2T^{n-2}\&c.$ and also in the 93d Proposition of the same Book, he gives the following expression, viz. $\overline{A+O}^{\frac{m}{n}} = A^{\frac{m}{n}} + \frac{m}{n}OA^{\frac{m-n}{n}} + \frac{m^2-mn}{2n^2}O^2A^{\frac{m-2n}{n}}$ &c.

De

De Moivres Multinomial Theorem will easily follow from the Binomial Theorem of Newton, which was given above: for, in $(Qz+Rz^2+Sz^3+Tz^4+\&c.)^{\frac{m}{r}}$ the co-efficient of $x^{\frac{m}{r}-n}$ is

$$\frac{\frac{m}{r}.(\frac{m}{r}-1).\&c.(\frac{m}{r}-\overline{n-1})}{1.\ 2.\ \&c..\ n}\times(Qz+Rz^2+Sz^3+Tz^4+\&c.)^n,$$

and in $(Qz+\overline{Rz^2+Sz^3+Tz^4+\&c.})^n$ the co-efficient of $(Qz)^{n-d}$ is

$$\frac{n.(n-1).\&c.(n-\overline{d-1})}{1.\ 2.\ \&c.\ d}\times(Rz^2+Sz^3+Tz^4+\&c.)^d$$

And in $(Rz^2+\overline{Sz^3+Tz^4+\&c.})^d$ the co-efficient of $(Rz^2)^{d-e}$ is

$$\frac{d.(d-1.)\&c.(d-\overline{e-1})}{1.\ 2.\ .\ \&c.\ e}\times(Sz^3+Tz^4+\&c.)^e$$

Therefore in $(x+\overline{Qz+Rz^2+Sz^3+Tz^4+\&c.})^{\frac{m}{r}}$

the co-efficient of $x^{\frac{m}{r}-n}\times(Qz)^{n-d}\times(Rz^2)^{d-e}\times(Sz^3)^{e-f}\times\&c.$ will be

$$\frac{\frac{m}{r}.(\frac{m}{r}-1).\&c.(\frac{m}{r}-\overline{n-1})}{1.2.3.\&c.\ n}\times\frac{n.(n-1).\&c.(n-\overline{d-1})}{1.2.3.\&c.\ d}\times$$

$$\frac{d.(d-1).\&c.(d-\overline{e-1})}{1.2.3.\&c.\ e}\times\frac{e.(e-1).\&c.(e-\overline{f-1})}{1.2.3.\&c.\ f}\times\&c.$$

Now, for $z^{n-d}\times z^{2(d-e)}\times z^{3(e-f)}\times\&c.$ put $z^{(n+d+e+f+\&c.)}$ and simplify the co-efficient, and you will have this general expression for any term:

$$\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).\&c.\ .\ .\ .\ .\ .\ \&c.(\frac{m}{r}-\overline{n-1})}{1.2..\&c.(n-d)\times(1.2..\&c.(d-e)\times1.2.\&c.(e-f)}\times$$

$$x^{\frac{m}{r}-n}\times Q^{n-d}\times R^{d-e}\times S^{e-f}\times\&c.\times z^{(n+d+e+f+\&c.)}$$

The

The multinomial Theorem has been generally expanded in the following form, viz.

$$(x+Qz+Rz^2+Sz^3+\&c.)^{\frac{m}{r}}=$$

$$x^{\frac{m}{r}}+\frac{m}{r}x^{\frac{m}{r}-1}Qz+\frac{\frac{m}{r}.(\frac{m}{r}-1)}{1\,.\,2}.x^{\frac{m}{r}-2}Q^2z^2+\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)}{1\,.\,2\,.\,3}x^{\frac{m}{r}-3}Q^3z^3+\&c.$$

$$+\frac{m}{r}x^{\frac{m}{r}-1}Rz^2 \quad +\frac{\frac{m}{r}.(\frac{m}{r}-1)}{1\times1}.x^{\frac{m}{r}-2}QRz^3+\&c.$$

$$+\frac{m}{r}x^{\frac{m}{r}-1}Sz^3 \qquad +\&c.$$

$$+\&c.$$

Now, in this expression, the first quantity that is expanded is $\overline{x+Qz}|^{\frac{m}{r}}$, and then $\frac{m}{r}\times\overline{x+Qz}|^{\frac{m}{r}-1}\times(Rz^2+Sz^3+\&c.)$, and afterwards $\frac{\frac{m}{r}.(\frac{m}{r}-1).}{1\,.\,2}\times\overline{x+Qz}|^{\frac{m}{r}-2}\times(Rz^2+Sz^3+\&c.)^2$, and so on.

But the arrangement which would follow from the Demonstration which I have given, would be the following, viz.

$$(x+Qz+Rz^2+Sz^3+\&c.)^{\frac{m}{r}}=$$

$$x^{\frac{m}{r}}+\frac{m}{r}x^{\frac{m}{r}-1}(Qz+Rz^2+Sz^3+\&c.)$$

$$+\frac{\frac{m}{r}.(\frac{m}{r}-1)x^{\frac{m}{r}-2}}{1.\;2}(Qz+Rz^2+Sz^3+\&c.)^2$$

$$+\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)x^{\frac{m}{r}-3}}{1.\;\;2.\;\;3}(Qz+Rz^2+Sz^3+\&c.)^3$$

$$+\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).(\frac{m}{r}-3)}{1.\;\;2.\;\;3.\;\;4}(Qz+Rz^2+Sz^3+\&c.)^4$$

$$+\&c.$$

According

According to this arrangement, the $\frac{m}{r}^{th}$ power of the above Multinomial, will be thus expanded, viz.

$$(x+Qz+Rz^2+Sz^3+\&c.)^{\frac{m}{r}}=$$

$$x^{\frac{m}{r}}+\tfrac{m}{r}x^{\frac{m}{r}-1}Qz+\tfrac{m}{r}x^{\frac{m}{r}-1}Rz^2+\tfrac{m}{r}x^{\frac{m}{r}-1}Sz^3+\tfrac{m}{r}x^{\frac{m}{r}-1}Tz^4+\&c.$$

$$+\frac{\frac{m}{r}.(\frac{m}{r}-1)}{1\,.\,2}x^{\frac{m}{r}-2}Q^2z^2+\frac{\frac{m}{r}.(\frac{m}{r}-1)}{1\times 1}x^{\frac{m}{r}-2}QRz^3+\frac{\frac{m}{r}.(\frac{m}{r}-1)}{1\times 1}x^{\frac{m}{r}-2}QSz^4$$

$$+\frac{\frac{m}{r}(\frac{m}{r}-1)}{1.2}x^{\frac{m}{r}-2}R^2z^4+\&c.$$

$$+\frac{\frac{m}{r}}{1}.\frac{\frac{m}{r}-1}{2}.\frac{\frac{m}{r}-2}{3}.x^{\frac{m}{r}-3}Q^3z^3+\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)}{1\,.\,2\times 1}x^{\frac{m}{r}-3}Q^2Rz^4+\&c.$$

$$+\frac{\frac{m}{r}}{1}.\frac{\frac{m}{r}-1}{2}.\frac{\frac{m}{r}-2}{3}.\frac{\frac{m}{r}-3}{4}.x^{\frac{m}{r}-4}Q^4z^4+\&c.$$

De Moivre has given the Multinomial Theorem in the Philosophical Transactions of 1697. His proof, however, from the doctrine of combinations extended only to integral powers which are produced by repeated multiplications; but Newton's Theorem having now been demonstrated, and the Multinomial Theorem having from thence been derived, it will follow that the latter Theorem is as general as the former, whether the

the Index of the power be an Integer, a Fraction, or even a Surd, as will easily appear from the observation which I have already made at the end of the former Chapter.

ESSAYS

ON

POWERS AND THEIR DIFFERENCES.

BY FRANCIS BURKE, Esq. &c. &c.

THE SECOND ALGEBRAICAL ESSAY.

On finding, per Saltum, *the several Orders of Differences.*

INTRODUCTION.

THE Formulæ for finding, *per Saltum*, the several orders of differences of quantities in a series, have been usually deduced from a repeated algebraical subduction. Thus, if the successive quantities are a, b, c, d, e, &c. the first differences are $a-b$, $b-c$, $c-d$, $d-e$, &c.; and the second differences, or the differences of the first differences, are, $a-2b+c$, $b-2c+d$, $c-2d+e$, &c.; and the third differences $a-3b+3c-d$, $b-3c+3d-e$, &c.; and, the coefficients being produced like the unciæ of the powers of $(1-x)^n$, the n^{th} differences are $a-nb+\frac{n.(n-1)}{1.2}c-\frac{n(n-1)(n-2)}{1.2.3}d+$&c, $b-nc+\frac{n.(n-1)}{1.2}d-\frac{n(n-1)(n-2)}{1.2.3}e$ +&c, &c. And if the quantities are a^m, b^m, c^m, d^m, &c. the

the n^{th} differences are found by the same operations, and are $a^m - nb^m + \frac{n.(n-1)}{1.2}c^m - \frac{n.(n-1).(n-2)}{1.2.3}d^m + \&c$, $b^m - nc^m + \frac{n.(n-1)}{1.2}d^m - \frac{n.(n-1).(n-2)}{1.2.3}e^m +$ &c, &c. in which the successive quantities in the series of powers are made also to enter into the expression for the differences of the powers. The method of Fluxions, although a particular case of the method of differences, cannot from the above Formulæ be immediately deduced, since those Formulæ suppose the successive quantities to be known: whereas, in the method of Fluxions, no account is to be taken of the successive values of variable quantities, and to express the several orders of Fluxions of powers, we have no other notation than by expressing them in terms of the Fluxions of the Roots. Therefore, putting Q instead of $a-b$, and R for $a-2b+c$, and, for the first of the third differences, or $a-3b+3c-d$ putting S, &c. &c., if we would represent in terms of these the n^{th} differences of the powers, or such parts of them as are constant, we would enlarge the analogy which has been observed, in some cases, to hold between differences and Fluxions. And this is the object of the following Essay, in which the 5th Proposition and the differential Problems will include the very useful and general Formula, for the co-efficients in the method of finding Fluxions per Saltum, as discovered by the Rev. Dr. Brinkley, the Professor of Astronomy, in the University of Dublin. See the 7th vol. of "The Transactions of the Royal Irish Academy," p. 327.

ESSAY

ESSAY II.

CHAP. I.

PROP. I.

AS a principle in the proof of the Binomial Theorem, it has been already demonstrated, that the n^{th} differences of the n^{th} powers of numbers in arithmetical progression will be constant and equal to the common difference of the Roots multiplied by the product of the digits from unity to the Index of the power. But, as this shall be a principle in the following Propositions, I shall demonstrate it here in more particular terms than could be admitted in the proof of the more general Proposition from which it was deduced as a Corollary.

Let p, q, r, s, t, &c. be the terms of an arithmetical series, whose common difference is d, then $q+d$, $r+d$, $s+d$, $t+d$, $v+d$, &c. will be respectively equal to the corresponding terms of the former series, p, q, r, s, t, &c. since each Binomial has for its first member the next lesser term of the series, and for its second member, the common difference; and the first members of the n^{th} powers of those Binomes being the n^{th}

powers

powers of the first members of the Roots, will be the n^{th} powers of the next lesser terms of the series, viz.

FIRST DIFFERENCES.

$$p^n = \overline{q+d}|^n = q^n + ndq^{n-1} + Cd^2q^{n-2} + \&c. \qquad ndq^{n-1} + Cd^2q^{n-2} + \&c.$$

$$q^n = \overline{r+d}|^n = r^n + ndr^{n-1} + Cd^2r^{n-2} + \&c. \qquad ndr^{n-1} + Cd^2r^{n-2} + \&c.$$

$$r^n = \overline{s+d}|^n = s^n + nds^{n-1} + Cd^2s^{n-2} + \&c. \qquad nds^{n-1} + Cd^2s^{n-2} + \&c.$$

$$s^n = \&c.$$

Hence, if we take away from the n^{th} powers of the Binomials the first members of those powers, we shall have taken from them the n^{th} powers of the next lesser terms of the arithmetical series, and the remainders are the first differences of the n^{th} powers of the Binomes. Therefore the $n-1^{th}$ differences of the remainders are the n^{th} differences of the n^{th} powers, but those remainders involve only lesser powers of the first members of the Binomes, which first members are in the same arithmetical series, and the highest of those powers are the $n-1^{th}$ powers of the terms of that arithmetical series, the coefficient of these powers being $n \times d$.

Therefore in the case where the Index n is equal to 2, the second differences of the squares will be common, for they are the first differences of the terms of an arithmetical series multiplied

multiplied by $2d$, the co-efficient of the second term. Therefore, the second differences of the squares $=d\times 2d$.

	FIRST DIFFERENCES.
$p^2=q^2+2dq+d^2$	
	$2dq+d^2$
$q^2=r^2+2dr+d^2$	
	$2dr+d^2$
$r^2=s^2+2ds+d^2$	
	$1ds+d^2$
$s^2=$&c.	

The third differences of the cubes will be the second differences of their first differences; i. e. of the remainders, after taking the highest members away. But substituting 3 for n, the second differences of the remainders $=3d$ multiplied into the second differences of the squares, since the second differences of the oneth powers $=0$. Therefore the third differences of the cubes $=d\times 2d\times 3d$.

	FIRST DIFFERENCES.
$p^3=q^3+3dq^2+3d^2q+d^3$	
	$3dq^2+3d^2q+d^3$
$q^3=r^3+3dr^2+3d^2r+d^3$	
	$3dr^2+3d^2r+d^3$
$r^3=s^3+3ds^2+3d^2s+d^3$	
	$3ds^2+3d^2s+d^3$
$s^3=$&c.	

In general, when the proposition is proved of all the powers whose Index is less than n, (as it has been proved of the second and third powers), in this manner also the case of the n^{th} powers will be deduced, viz. The $n-1^{th}$ differences of the

the remainders, or the n^{th} differences of the given powers, will be the $\overline{n-1}^{th}$ differences of n times the $\overline{n-1}^{th}$ powers, which are the highest members in those remainders; for the $\overline{n-1}^{th}$ differences of the members containing lower powers, will be the differences of common differences, and therefore $=0$. If then (as we have shewn in the second and third powers) the $\overline{n-1}^{th}$ differences of the $\overline{n-1}^{th}$ powers be $=1d.2d...\overline{n-1}d$, the n^{th} differences of the n^{th} powers $=1d.2d...\overline{n-1}d\times nd$ or $=1.2.3...\overline{n-1}.n\times d^n$.

PROP. II.

If there are n number of arithmetical series, whose common differences are respectively a, b, c, d, &c. let all the different corresponding terms of the several series be multiplied together, the n^{th} differences of the products will be constant and equal to $1.2.3...n\times abcd$ &c.

For, let A, B, C, D, &c. be the correspondent terms of the different series, the products will then be

$$\begin{array}{l} \overline{3a+A}\times\overline{3b+B}\times\overline{3c+C}\times\overline{3d+D}\times\&c. \\ \overline{2a+A}\times\overline{2b+B}\times\overline{2c+C}\times\overline{2d+D}\times\&c. \\ \overline{a+A}\times\overline{b+B}\times\overline{c+C}\times\overline{d+D}\times\&c. \\ A\times B\times C\times D\times\&c. \end{array}$$

And

And those factors being multiplied, the products will be

$3^n \times abcd$ &c. $+3^{n-1} \times \overline{Abc \text{ \&c. } + aBc \text{ \&c.}}$ + &c.+ABCD &c.

$2^n \times abcd$ &c. $+2^{n-1} \times \overline{Abc \text{ \&c. } + aBc \text{ \&c.}}$ +&c.+ABCD &c.

$1^n \times abcd$ &c. $+1^{n-1} \times \overline{Abc \text{ \&c. } + aBc \text{ \&c.}}$ +&c.+ABCD &c.

+ABCD &c.

The highest terms in those products are the n^{th} powers of the natural numbers with the constant co-efficient *abcd* &c which is the product of all the common differences. The subsequent terms are the inferior powers of the natural numbers, with certain constant co-efficients. But in the n^{th} differences, all the inferior powers are exterminated, the terms which are the differences of common differences being =0. Therefore, the n^{th} differences of the contents are the n^{th} differences of their highest members, or the n^{th} differences of the n^{th} powers of the natural numbers with the constant co-efficient, *abcd* &c. or $=1.2.3 \ldots n \times abcd$ &c.

PROP. III.

The series $1, \frac{n+1}{1}, \frac{(n+1).(n+2)}{1\,.\,2}, \frac{(n+1).(n+2\,(n+3)}{1\,.\,2\,.\,3}$, &c. will have the n^{th} differences of its terms common and equal to unit, (which is the definition of triangular numbers of the n^{th} order): For

For if we take as an example of the preceding proposition, the contents, &c × &c × &c..&c.

$$\begin{array}{l} 4 \times 5 \times 6 \ldots (n+3) \\ 3 \times 4 \times 5 \ldots (n+2) \\ 2 \times 3 \times 4 \ldots (n+1) \\ 1 \times 2 \times 3 \ldots n \end{array}$$

in which the different arithmetical progressions are the natural series, and also A, B, C, D, or the corresponding terms are the natural series 1, 2, 3, &c. to n. One of the contents $= 1.2.3 \ldots n$, which is equal to the n^{th} differences of the contents. Therefore, if we divide the quantities $1.2 \ldots n$, $2.3 \ldots (n+1)$, $3.4 \ldots (n+2)$, $4.5 \ldots (n+3)$, &c. by $1.2 \ldots n$ unity will be equal to the n^{th} differences of the quotients 1, $\frac{n+1}{1}$, $\frac{(n+1).(n+2)}{1\ .\ 2}$, $\frac{(n+1).(n+2).(n+3)}{1\ .\ 2\ .\ 3}$, &c.

PROP. IV.

If the q^{th} powers of the natural numbers, and the r^{th} powers of the corresponding triangular numbers which admit two orders of differences, and the corresponding s^{th} powers of the triangular numbers having three orders, &c. be all multiplied as follows: $1^q \times 1^r \times 1^s \times$ &c.

$$\begin{array}{l} 2^q \times 3^r \times 4^s \times \&c. \\ 3^q \times 6^r \times 10^s \times \&c. \\ \&c. \times \&c. \times \&c. \times \&c. \end{array}$$

And

And if we take the differences of those products, they will have as many orders of differences as $q+2r+3s+\&c.$ and the last of those will be constant and $=\dfrac{1\times 2\times 3\times 4\times \&c.\ldots(q+2r+3s+\&c.)}{1^{q}\times\overline{1.2}|^{r}\times\overline{1.2.3}|^{s}\times\&c.}$

For, the several powers of contents being multiplied by each other, in the horizontal lines in which they correspond together, viz.

$$\text{The } q^{\text{th}} \text{ powers of the contents,}\left\{\begin{array}{l}1.2.3\ldots g\\ 2.3.4\ldots(g+1)\\ 3.4.5\ldots(g+2)\\ \&c.\end{array}\right\}\begin{array}{c}\text{Multiplied by the } r^{\text{th}}\\ \text{powers of the cor-}\\ \text{responding contents}\end{array}\left\{\begin{array}{l}1.2.3\ldots h\\ 2.3.4\ldots(h+1)\\ 3.4.5\ldots(h+2)\\ \&c.\end{array}\right\}\begin{array}{c}\text{And by the } s^{\text{th}} \text{ powers}\\ \text{of the contents,}\end{array}\left\{\begin{array}{ll}1.2.3\ldots k & \times\&c.\\ 2.3..(k+1) & \times\&c.\\ 3.4..(k+2) & \times\&c.\\ \&c. & \times\&c.\end{array}\right.$$

the results of the multiplication will be as follows,

$$\left\{\begin{array}{lllll}1.2\ldots.g\times 1.2\ldots g\times\&c. & \text{till repeated } q \text{ times,} & 1.2\ldots h\times\&c. & \text{till repeated } r \text{ times,} & \\ 2.3\ldots(g+1)\times 2.3\ldots(g+1) & \times\&c\cdots\cdots & 2.3..(h+1) & \times\&c\cdots\cdots & \\ 3.4\ldots(g+2)\times 3.4\ldots(g+2) & \times\&c\cdots\cdots & 3.4\ldots(h+2) & \times\&c\cdots\cdots & \\ \&c. \quad\times\&c. & \times\&c\cdots\cdots & \&c. & \times\&c\cdots\cdots & \end{array}\right.$$

$$\left.\begin{array}{lllll}\times 1.2.3\ldots.k & \times 1.2.3\ldots k & \times\&c. & \text{Till repeated } s \text{ times,} & \times\&c.\\ \times 2.3.4\ldots(k+1) & \times 2.3.4\ldots(k+1) & \times\&c.\cdots\cdots\cdots\cdots & & \times\&c.\\ \times 3.4.5\ldots(k+2) & \times 3.4.5\ldots(k+2) & \times\&c.\cdots\cdots\cdots\cdots & & \times\&c.\\ \times\&c. & \times\&c. & \times\&c.\cdots\cdots\cdots\cdots & & \times\&c.\end{array}\right\}$$

In which the Involution producing a repetition of contents, each of which has several factors, the series resulting from the multiplication of the powers of the corresponding contents

will now be the contents of as many arithmetical series as $gq+hr+ks+\&c.$ which sum putting equal to n, the n^{th} differences of the products of the corresponding terms of n arithmetical series, which (by the second Prop.) $=1.2.3...n$, will be the n^{th} differences of the products of the powers of the contents. But, (from Prop. 3.) if we divide those products of the powers of the contents by $\overline{1.2.3...g}|^q \times \overline{1.2.3...h}|^r \times \overline{1.2.3...k}|^s + \&c.$ we shall have the products of the powers of the corresponding terms of the different series of triangular numbers, whose orders are g, h, k, &c. Therefore the n^{th} differences of these will be $1.2.3...n$, divided by $\overline{1.2.3...g}|^q \times \overline{1.2.3...h}|^r \times \overline{1.2.3...k}|^s \times$ &c. $\because = \dfrac{1.2.3.4\ldots\ldots n}{\overline{1.2.3...g}|^q \times \overline{1.2.3...h}|^r \times \overline{1.2.3...k}|^s \times \&c.}$

LEMMA.

The quantities in any series can be expressed in a multinomial form in terms of triangular numbers and of the first of the several orders of differences of the quantities: viz. let P be the first quantity taken in a series, and let Q, R, S, &c. be the first of the several differences whose orders are one, two, three, &c.; the following general formulæ will express the preceding and subsequent quantities in the series, viz.

GENERAL

GENERAL FORMULÆ.

$$P+nQ+\frac{n.(n+1)}{1.2}R+\frac{n.(n+1).(n+2)}{1.2.3}S\ \&c.=n^{th}\ \text{preceding quantity}$$

$$\begin{array}{llllllll}
\&c.+\&c.+ & \&c. & + & \&c. & +\&c. & +\&c. & & \\
P+3Q+ & 6R & + & 10S & +\&c. & +\frac{(n+1).(n+2)}{1\ .\ 2}D_n & +\&c. & \\
P+2Q+ & 3R & + & 4S & +\&c. & +(n+1)D_n & +\&c. & \\
P+Q\ \ + & R & + & S & +\&c. & +D & +\&c. &
\end{array}$$

$P=$ the first of the given quantities.

$$\begin{array}{lllll}
P-Q & & & & \\
P-2Q\ + & R & & & \\
P-3Q\ + & 3R & - & S & \\
\&c.-\&c.+ & \&c. & - & \&c. & +\&c.
\end{array}$$

$$P-nQ+\frac{n.(n-1)}{1.2}R-\frac{n.(n-1).(n-2)}{1.2.3}S\ \&c.=n^{th}\ \text{succeeding quantity.}$$

For, since in those formulæ, the co-efficients of Q have their differences equal to unit, and the co-efficients of R, their second differences, $=1$, &c. if for P, Q, R, &c. in the formulæ, we substitute Q, R, S, &c. respectively, we shall have the first differences of the quantities; and by substituting R, S, T, &c. we have the second differences, &c. Therefore, P being

the first quantity of the series of assumed formulæ, Q is the first of the differences of the first order, R, the first of the second order of differences of the assumed quantities, &c. Therefore, the law of the assumed formulæ is the same with that of the series whose orders began with Q, R, S, &c. Therefore all the formulæ in the series are rightly expressed as well as P.

NOTE. When the given series has no constant order of differences, the expression for preceding quantities, viz. $P+Q+R+S+$&c. will be an infinite series, I have therefore given the above proof, by shewing that the differences of the assumed, and those of the given quantities, are identical, since the proof which is usually given by a summation beginning from the last order of differences, can only be applied where there is such a constant order of differences to be found.

PROP. V.

If P is one of the quantities in a series admitting several orders of differences, the first of the differences in the first, second, third, &c. orders of differences of those quantities being Q, R, S, &c. putting $(p-d)+(d-e)+$&c.$=p$, and $(p-d)+2(d-e)+3(e-f)+$&c. or $p+d+e+f+$&c.$=n$. In all the n^{th} differences of the $\frac{m}{r}^{th}$ powers of those quantities in the Lemma, the co-efficient of $P^{\frac{m}{r}-p}\times Q^{p-d}\times R^{d-e}\times S^{e-f}\times$&c. will

will be constant or the same in all the differences of that order, and will be =

$$\frac{1.2.3.4\ldots\ldots n}{1^{p-d}\times\overline{1.2}^{d-e}\times\overline{1.2.3}^{e-f}\times\&c.}\times\frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)\ldots(\frac{m}{r}-\overline{p-1})}{1.2.3\ldots\overline{p-d}\times 1.2.3\ldots\overline{d-e}\times 1.2.3\ldots\overline{e-f}\times\&c.}$$

For, if we expand the $\frac{m}{r}^{th}$ powers of the polynomial quantities in the above Lemma, by the multinomial theorem,

In $(P+Q+R+S+\&c.)^{\frac{m}{r}}$

we have $P^{\frac{m}{r}-p}\times Q^{p-d}\times R^{d-e}\times S^{e-f}\times\&c.$

with a co-efficient $\dfrac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).(\frac{m}{r}-3)\ldots(\frac{m}{r}-\overline{p-1})}{1.2.3\ldots\overline{p-d}\times 1.2.3\ldots\overline{d-e}\times 1.2.3\ldots\overline{e-f}\times\&c.}$

Also in $(P+2Q+3R+4S+\&c.)^{\frac{m}{r}}$

we have $P^{\frac{m}{r}-p}\times 2^{p-d}Q^{p-d}\times 3^{d-e}R^{d-e}\times 4^{e-f}S^{e-f}\times\&c.$

with the same co-efficient.

And in $(P+3Q+6R+10S+\&c.)^{\frac{m}{r}}$

we have $P^{\frac{m}{r}-p}\times 3^{p-d}Q^{p-d}\times 6^{d-e}R^{d-e}\times 10^{e-f}S^{e-f}\times\&c.$

with the same co-efficient, &c.

But by the preceding proposition the n^{th} differences of

$1^{p-d}\times 1^{d-e}\times 1^{e-f}\times\&c.$

$2^{p-d}\times 3^{d-e}\times 4^{e-f}\times\&c.$

$3^{p-d}\times 6^{d-e}\times 10^{e-f}\times\&c.$

&c.

will be constant and equal to $\dfrac{1\;.\;2\;.\;3\;.\;.\;.\;.\;n}{1^{p-d}\times\overline{1.2}^{d-e}\times\overline{1.2.3}^{e-f}\times\&c.}$

Therefore,

Therefore, in the n differences of the $\frac{m}{r}$ powers, we have $P^{\frac{m}{r}-p} \times Q^{p-d} \times R^{d-e} \times S^{e-f} \times$ &c. with the constant co-efficient

$$\frac{1 \,.\, 2 \,.\, 3 \,.\,.\,.\,.\,.\, n}{1^{p-d} \times \overline{1.2}^{d-e} \times \overline{1.2.3}^{e-f} \times \&c.} \times \frac{\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2) \,.\,.\,.\,.\, (\frac{m}{r}-\overline{p-1})}{1.2.3..\overline{p-d} \times 1.2.3...\overline{d-e} \times 1.2.3..\overline{e-f} \times \&c.}$$

The several formulæ laid down in the following Chapter are deduced from Prop. 4. and the Lemma, in like manner as the general co-efficient now given, by means of the multinomial theorem, which I have stated in the first Essay, wherein the unciæ of powers were deduced from the differences of powers of numbers, which, having a common difference, are in an arithmetical progression.

CHAP.

CHAP. II.

Problems for finding, per Saltum, all the constant parts in the Differences of any order.

PROB. I.

ALL the constant parts in the n^{th} differences of the $\frac{m}{r}^{th}$ powers of quantities not increasing uniformly, will be

$$\frac{m}{r}P^{\frac{m}{r}-1}D_{n} + \left.\begin{array}{l} \dfrac{1.2.3\ldots n}{1\times 1.2.3\ldots(n-1)}QD_{n-1} \\ +\dfrac{1.2.3\ldots n}{1.2\times 1.2.3\ldots(n\text{-}2)}\times RD_{n-2} \\ +\&c. \end{array}\right\}\times\frac{m}{r}.\left(\frac{m}{r}-1\right)P^{\frac{m}{r}-2}$$

$$\left.\begin{array}{l} +\dfrac{1.2.3\ldots n}{1.2\times 1.2.3\ldots(n-2)}QD_{n-2} \\ +\dfrac{1.2.3\ldots n}{1\times 1.2\times 1.2.3\ldots(n-3)}QRD_{n-3} \\ +\&c. \end{array}\right\}\times\frac{m}{r}.\left(\frac{m}{r}-1\right).\left(\frac{m}{r}-2\right)P^{\frac{m}{r}-3}+\&c.$$

which may be also expressed in the following form, by beginning with the constant parts in which P has the lowest index, viz. $\frac{m}{r}.\left(\frac{m}{r}-1\right)\ldots\left(\frac{m}{r}-\overline{n-1}\right)P^{\frac{m}{r}-n}Q^{n}$

$$+\frac{n(n-1)}{1.2}\frac{m}{r}.\left(\frac{m}{r}-1\right)\ldots\left(\frac{m}{r}-\overline{n-2}\right)P^{\frac{m}{r}-\overline{n-1}}\times Q^{n-2}\times R$$

$$+\frac{n.(n-1).(n-2)}{1.2.3}\frac{m}{r}.\left(\frac{m}{r}-1\right)\ldots\left(\frac{m}{r}-\overline{n-3}\right).P^{\frac{m}{r}-\overline{n-2}}\times Q^{n-3}\times S+\&c.$$

the

the diversity of form arising only from the different arrangement of the terms in the $\frac{m}{r}^{th}$ powers of the multinomials.

Demonstration. The $\frac{m}{r}^{th}$ powers of the expressions deduced from the Lemma (the co-efficients being understood) will be of this form, viz.

$$
\begin{array}{l}
P^{\frac{m}{r}} \\
+QP^{\frac{m}{r}-1}+RP^{\frac{m}{r}-1}+SP^{\frac{m}{r}-1} \quad +TP^{\frac{m}{r}-1} \qquad +VP^{\frac{m}{r}-1} \qquad +\&c. \\
\qquad Q^2P^{\frac{m}{r}-2}+QRP^{\frac{m}{r}-2}+(QS+R^2)P^{\frac{m}{r}-2}+(QT+RS)P^{\frac{m}{r}-2}+\&c. \\
\qquad\qquad +Q^3P^{-3} \quad +Q^2RP^{\frac{m}{r}-3} \quad +(Q^2S+QR)P^{\frac{m}{r}-3}+\&c. \\
\qquad\qquad\qquad +Q^4P^{\frac{m}{r}-4} \qquad +Q^3RP^{\frac{m}{r}-4} \qquad +\&c. \\
\qquad\qquad\qquad\qquad +Q^5P^{\frac{m}{r}-5} \qquad +\&c.
\end{array}
$$

$$
\begin{array}{l}
+D_nP^{\frac{m}{r}-1} \qquad\qquad\qquad\qquad +\&c. \\
+(QD_{n-1}+RD_{n-2} \quad +SD_{n-3} \quad +\&c.)\times P^{\frac{m}{r}-2} \qquad +\&c. \\
+(Q^2D_{n-2}+QRD_{n-3}+R^2D_{n-4} \quad +\&c.)\times P^{\frac{m}{r}-3} \qquad +\&c. \\
+(Q^3D_{n-3}+Q^2RD_{n-4}\div Q^2SD_{n-5}+QR^2D_{n-5}+\&c.)\times P^{\frac{m}{r}-4}+\&c. \\
+(Q^4D_{n-4}+Q^3RD_{n-5}+\&c.) \qquad \times P^{\frac{m}{r}-5} \qquad +\&c.
\end{array}
$$

By prefixing to the powers of the letters Q, R, S, &c. the same powers of their variable co-efficients, or of the triangular numbers

numbers, and prefixing the constant co-efficients of the terms in the $\frac{m}{r}^{th}$ powers of a multinomial, we shall have the $\frac{m}{r}^{th}$ powers of the successive quantities in the Lemma; in the n^{th} differences of these, all the terms are exterminated, in which, such powers of the triangular numbers are multiplied together, that the number of orders which the triangular numbers admit, being multiplied by the indices of the powers, and the products being added together, the aggregate is less than n, viz. when the indices $p-d$, $d-e$, $e-f$, &c. are such that $(p-d) + 2(d-e) + 3(e-f) +$ &c. is less than n, the co-efficients of $Q^{p-d} \times R^{d-e} \times S^{e-f} \times$ &c. will in the n^{th} differences be exterminated; but when that sum is equal to n, the co-efficients of those quantities will be constant, and are known from Prop 5.

Hence, all that can be constant in the first differences will be $\frac{m}{r}P^{\frac{m}{r}-1}Q$; in the second differences, $\frac{m}{r}P^{\frac{m}{r}-1}R + Q^2\frac{m}{r}.(\frac{m}{r}-1)P^{\frac{m}{r}-2}$; the constant members found in the third differences will be,

$$\frac{m}{r}P^{\frac{m}{r}-1}S + \frac{3.2.1}{1\times 1.2}QR\frac{m}{r}.(\frac{m}{r}-1)P^{\frac{m}{r}-2} + \frac{3.2.1}{1.2\,3}Q^3\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)P^{\frac{m}{r}-3}$$

the constant members found in the fourth differences will be,

$$\frac{m}{r}P^{\frac{m}{r}-1}T + \left.\begin{matrix}\frac{4.3.2.1}{1\times 1.2\,3}QS \\ + \frac{4.3\,2.1}{1.2\times \overline{1\,2}|^2}R^2\end{matrix}\right\}\frac{m}{r}.(\frac{m}{r}-1)P^{\frac{m}{r}-2} + \frac{4\,3.2.1}{1.2\times 1.2}Q^2R\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)P^{\frac{m}{r}-3}$$

$$+\frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2).(\frac{m}{r}-3)P^{\frac{m}{r}-4}Q^4$$

and so on, for the higher orders of differences.

PROB. II.

If in a series of Quotients one of which is $\frac{p}{P}$ the dividends admit several orders of differences, of which the differences corresponding to p in the several orders, are $q, r, s, t,$ &c. d_n, &c. and in like manner, if of the differences of the divisors, those which in the several orders correspond to P, are Q, R, S, T, &c. D_n, &c. Then, all the constant parts in the n^{th} differences of the terms of the series of quotients are,

$$\left.\begin{array}{l} d_n P^{-1} + \frac{1.2.3...n}{1.2.3...n} p D_n \\ \quad + \frac{1.2.3...n}{1 \times 1.2.3...(n-1)} q D_{n-1} \\ \quad + \&c. \end{array}\right\} \times -1 \times P^{-1-1}$$

$$\left.\begin{array}{l} + \frac{1.2.3...n}{1 \times 1.2.3...(n-1)} p Q D_{n-1} \\ + \&c. \end{array}\right\} \times -1 \times (-1-1) P^{-1-2} + \&c.$$

For, if the -1^{th} power of $\overline{P+Q+R+S+\&c.}$ (which is of the same form as above, only substituting -1 for $\frac{m}{r}$) be multiplied by $\overline{p+q+r+s+\&c}$, and the several multiplications of the terms be performed, according to the powers of P, taking also into account the places of the members of the multinomial multiplier. In the place of the quantities included in the several lines in the proof of the preceding Problem, we shall now have the quantities, in the following expression, contained

contained in the several braces, and the quote (without regarding the co-efficients) is of this form,

$$(p+q+r+s+t \quad +\&c.+d_n \quad +\&c.)P^{-1}$$

$$\left\{\begin{array}{llll} pq+pR+pS & +pT & +\&c.+pD_n & +\&c. \\ \quad +qQ+qR & +qS & +\&c.+qD_{n-1} & +\&c. \\ \qquad +rQ & +rR & +\&c.+rD_{n-2} & +\&c. \\ & +sQ & +\&c.+sD_{n-3} & +\&c. \\ & & \quad +\&c. & +\&c. \end{array}\right\} P^{-1-1}$$

$$\left\{\begin{array}{lll} pQ^2+pQR+p(QS+R^2)+\&c.+p(QD_{n-1}+RD_{n-2}+\&c.)+\&c. & & \\ \quad +qQ^2 \quad +qQR \qquad +\&c.+q(QD_{n-2}+RD_{n-3}+\&c.)+\&c. & & \\ \qquad\qquad +rQ^2 \qquad\quad +\&c.+r(QD_{n-3}+RD_{n-4}+\&c.)+\&c. & & \\ \qquad\qquad\qquad\qquad +\&c. & & \end{array}\right\} P^{-1-2}$$

$$\left\{\begin{array}{lll} +pQ^3 & +pQ^2R & +\&c.+p(Q^2D_{n-2}+QRD_{n-3}+\&c.)+\&c. \\ & +qQ^3 & +\&c.+q(Q^2D_{n-3}+QRD_{n-4}+\&c.)+\&c. \\ & & \quad +\&c. \qquad\qquad\qquad +\&c. \end{array}\right\} P^{-1-3}$$

If d_v is put for any of the differences, q, r, s, &c. or in general for the first among the v^{th} differences of the dividends, d_v being always of one dimension, and its order being v, the product of its Index and order is v; therefore, putting $v+(p-d)+2(d-e)+3(e-f)+\&c.$ or $v+p+d+e+\&c.=n$, when the constant co-efficients of the -1^{th} power are expressed, and when each quote is represented in the above form, by ex-

 pressing

pressing the powers of the triangular numbers which are the variable co-efficients of d_v, Q, R, &c. as in Prop. 5, the following quantity will be found, viz.

$$\frac{n.(n-1).(n-2)\ldots 3.2.1}{1.2\ldots v\times\overline{1}^{\,p-d}\times\overline{1.2}^{\,d-e}\times\overline{1.2.3}^{\,e-f}\times\&c.}\times\frac{-1.-2.-3\ldots-p}{1.2.3\ldots\overline{p-d}\times 1.2.3\ldots\overline{d-e}\times\&c.}$$

$\times P^{-1-p}\times d_v\times Q^{p-d}\times R^{d-e}\times\&c.$ will be constant in all the n^{th} differences of the quotes. Thus, the several quantities of this kind which will be constant in all the 4^{th} differences of the terms of the series of quotients, are viz.

$$tP^{-1}\left.\begin{array}{l}-pT\\-4qS\\-6rR\\-4sQ\end{array}\right\}\times P^{-2}\left.\begin{array}{l}+(8pQS+6pR^2)\\+24qQR\\+12rQ^2\end{array}\right\}\times P^{-3}\left.\begin{array}{l}-36pQ^2R\\-21Q^3\end{array}\right\}\times P^{-4}+24pQ^4P^{-5}$$

PROB. III.

Let there be any series of products, the successive factors admitting several orders of differences, viz. Let one of the products be $A\times B\times C\times D\times$ &c. and in the series of factors to which A belongs, let the differences corresponding to A, in the several orders, be a_1, a_2, a_3, &c. a_n, &c. And of the series of factors to which B belongs, let the differences corresponding to B, in the several orders, be b_1, b_2, b_3, &c. b_n, &c. And in like manner, also, of the series of factors to which C belongs, let the differences corresponding in the several orders

orders be c_1, c_2, c_3, &c. c_n, &c. &c. All the constant parts in the n^{th} differences of the contents will be, viz.

$$(a_n \text{BCDE \&c.} + \text{A}b_n\text{CDE \&c.} + \text{AB}c_n\text{DE \&c.} + \text{\&c.})$$
$$+ n(a_{n-1}b_1\text{CDE \&c.} + a_1 b_{n-1}\text{CDE \&c.} + \text{\&c.})$$
$$+ \frac{n.n-1}{1.2}(a_{n-2}b_2\text{CDE \&c.} + a_2 b_{n-2}\text{CDE \&c.} + \text{\&c.})$$
$$+ \text{\&c.}$$
$$+ n.(n-1)\ (a_{n-2}b_1c_1\text{DE \&c.} + a_1b_{n-2}c_1,\ \text{DE \&c.} + \text{\&c.}) + \text{\&c.}$$
$$+ \frac{n.(n-1)\ (n-2)}{1 \times 1.2}(a_{n-3}b_2c_1\text{DE \&c.} + \text{\&c.} \qquad + \text{\&c.}$$
$$+ \text{\&c.} \qquad + \text{\&c.}$$

For, instead of the powers in Prop. V. we shall now have a series of products in which the factors are of the forms $\text{A} + a_1 + a_2 + \text{\&c.}$ $\text{B} + b_1 + b_2 + \text{\&c.}$ $\text{C} + c_1 + c_2 + \text{\&c.}$ and in which products, from the nature of multiplication, the combinations of differences are multiplied into all the other first terms but their own, &c. And, of the triangular numbers, which in the successive contents of this form, are the co-efficients of those differences, a_1, a_2, &c. b_1, b_2, &c. &c. whenever the combinations are such, that the sum of the orders of differences which the triangular numbers admit is equal to n, the n^{th} differences of the members containing these combinations will be constant. And if $g+h+i+\text{\&c.}=n$, in all the n^{th} differences we have the following quantity, viz.

$$\frac{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ .\ .\ .\ .\ .\ n}{1.2\ 3 \ldots g \times 1\ 2.3 \ldots h \times 1.2\ 3 \ldots i \times \text{\&c.}} \times (a_g b_h c_i \text{\&c} \times \text{ZYX\&c} + \text{ABC\&c} \times z_g y_h x_i \text{\&c} + \text{\&c})$$

NOTE.

NOTE. Besides the constant quantities expressed in the above several formulæ, the actual differences will consist also of certain variable quantities, except in the following cases, in which the actual differences themselves will be constant.

PROB. IV.

Let the greatest number of orders which a series will admit be g, and let the constant differences be a. Let the greatest number which another series will admit be h, and constant difference be b_h. And let another series admitting i orders have a constant difference c_i. &c. Then if $g+h+i+\&c=n$, of the contents of the corresponding terms in all those series the actual differences of the n^{th} order will be constant quantities, and are $=\dfrac{1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,.\,n}{1.2.3....g \times 1.2.3...h \times 1.2.3...i \times \&c.} \times a_g \times b_h \times c_i \times \&c.$ For, in the contents of the form $(A+\&c.+a_g) \times (B+\&c.+b_h)$ $(C+\&c.+c_i) \times \&c.$ the product of all the constant differences, or $a_g \times b_h \times c_i \times \&c.$ will be the common co-efficients of the products of the highest orders of the triangular numbers in the successive expressions deduced from the Lemma, and the n^{th} differences of the products of those numbers are constant by Prop. IV. But the n^{th} differences of the terms containing all except the highest orders are exterminated, the sum of the orders

orders of the factors being less in those terms than n, therefore the n^{th} differences of the contents will be the n^{th} differences of their highest members, and therefore equal to

$$\frac{1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,.\,n}{1.2.3..g \times 1.2.3..h \times 1.2.3..i \times \&c.} \times a_g \times b_h \times c_i \times \&c.$$

PROB. V.

Also, if any series have m number of orders, let d be the last difference, the n powers of the quantities of that series will have as many orders as $n \times m$, and the constant differences $= \frac{1.2.3...(mn-1).mn}{\overline{1\,.\,2\,.\,3\,.\,.\,.\,m}|^n} \times d^n$. For here $g+h+i+\&c. = m \times n$, and $a_g \times b_h \times c_i \times \&c. = d \times d \times \&c. = d^n$

EXAMPLE.

If of the roots, the differences of the second order be all equal to R, the cubes have their sixth order constant and $= \frac{1.2.3.4.5.6}{\overline{1.2}|^3} \times R^3 = 90R^3$. And this would follow from Problem the first, for in the sixth differences of the $\frac{m}{r}^{th}$ powers, we have the following term, viz. $\frac{1.2.3.4.5.6}{\overline{1.2}|^3 \times 1.2.3} R^3 \times \frac{m}{r}.(\frac{m}{r}-1).(\frac{m}{r}-2)P^{\frac{m}{r}-3}$ and (if $\frac{m}{r}=3$, and third and successive differences of the root $=0$) $=90R^3$.

If

If for the powers of Q, R, S, &c. we substitute the powers of the first, second, third, &c. fluxions of x, the above will be the formulæ for the several orders of fluxions of $x^{\frac{m}{r}}$. For, the fluxions of x are parts of the actual differences, no notice of the remaining parts being necessary to be taken. And if, in the formulæ for the differences of the powers, we substitute for the difference of the roots the fluxion of the root + the rejected part of the actual difference of the root, the powers of the fluxion of the root will be similar to those of the differences of the root, with the same co-efficients; the remaining powers of the fluxion of the root are not to be noticed. as involving the members rejected in deriving the fluxion of x from its actual differences.

ESSAYS

ON

POWERS AND THEIR DIFFERENCES.

BY FRANCIS BURKE, Esq. &c. &c.

THE THIRD ALGEBRAICAL ESSAY.

On finding Divisors of Equations.

INTRODUCTION.

SIR I. NEWTON has given no proof of the method of finding divisors, which he has delivered in his "Universal Arithmetic." But, in his second example for finding a binomial divisor, the proposed is $6y^4—y^3—21y^2+3y+20$, and the quantity with which the division is to be tried, is $y+\frac{4}{3}$, or, which he says is the same thing, $3y+4$. And here Saunderson supposed, that the expression of the general form $x+\frac{e}{f}$ was adopted by Newton, lest the divisor $fx+e$ should admit a simple divisor, or its terms admit a common measure. But this supposition cannot hold when the rule has set out with supposing the given quantity to have been previously divided

by all its simple divisors, in which case no compound divisor can have a simple divisor, for such would be also a simple divisor of the quantity proposed. If, as in the example $2x—b$, Newton had merely considered the quantity in the relation of a divisor, he would not first represent the divisor in the form $x—\frac{b}{2}$ and thence $2x—b$. For, from division alone it would not follow, that, if the sub-multiple found should divide the proposed, so should also the multiple. Therefore, Newton, in the examples $y+\frac{4}{3}$, $a—\frac{5}{6}$, &c. must have deduced those expressions from the nature of fractional roots, as they enter the factors of an original equation, where from $y+\frac{4}{3}=0$ follows $3y+4=0$. The case of a divisor of one dimension, in which the co-efficient of the highest term is unit, has been proved from the nature of equations by M'Laurin. In the following Chapter, I shall give a demonstration for every case, by distinguishing between integral, fractional, and surd roots.

If an equation admits of no rational divisor of one or of two dimensions, the rules which have been usually given are inadequate to discover a binomial or trinomial divisor which might serve for investigating roots. But since equations of higher than two dimensions may often appear as trinomials, of a form which is similar to that of quadratic equations, and thus be capable of a similar resolution, a statement of the rules which would extend to the finding of such divisors may be practically advantageous.

Besides

Besides, if it happens that several binomials or trinomials are only apparently deducible by the method, we may find a criterion for discovering those which shall not really succeed in the division, if we try whether the quote, which should result from that division, is discoverable in the same manner as the divisor. For this purpose it will be of use to discover a general rule for a polynomial divisor, and the series of numbers which apparently gives us the divisor, will point out to us the series of factors which should give us the quote; from whence, by the method of differences, or a numerical subduction, we shall avoid the repeated operations of a trial by algebraical division.

These, and some other advantages, are the practical objects of this Essay, for which the Reader is referred to the second Chapter. In the first Chapter, the general theory of the rule is laid down: it is deduced from the nature of an equation, which is a different view of the subject from that which is usually given by authors.

CHAP. I.

The general Theory from whence the Rules for finding Divisors of Equations are deduced.

If there is given any equation, as $mx^7+Px^6+Qx^5+Rx^4+Sx^3+Tx^2+Vx+W=0$, by substituting for the unknown letter x, any given quantity A, we shall have a result which is the absolute term of a transformed equation, whose roots are individually those of the given equation diminished by A. For, if we substitute $y+A$ for x, we shall have $m(y+A)^7+P(y+A)^6+Q(y+A)^5+R(y+A)^4+S(y+A)^3+T(y+A)^2+V(y+A)+W=0$ and when the powers of the binomials are expanded, the last terms of those powers will be the similar powers of A, and will contain no dimensions of y. Hence $mA^7+PA^6+QA^5+RA^4+SA^3+TA^2+VA+W$, the sum of those members which contain no dimension of y, will be the absolute term in an equation in which $y+A=x$ and $y=x-A$.

If for the unknown letter there be severally substituted the terms of a decreasing arithmetical series, 3, 2, 1, 0, -1, -2, -3, the results are the several absolute terms of transformed equations, whose roots are those of the given equation, respectively diminished by the substituted quantities; and since the

the roots are successively diminished by quantities in a decreasing arithmetical series, they are successively diminished by lesser numbers, and therefore in each transformed equation the roots will be greater than in the preceding, by the common difference of the substituted numbers. But where the highest term of the equation has a co-efficient, there are fractional roots, whose denominators, if rational, are some of the numeral divisors of m. For, the product of the roots with the signs changed is $\frac{W}{m}$; and since those fractional roots must differ in the successive transformed equations by the common difference of the substituted numbers, if the difference of the substituted numbers be reduced to the same denominators with the fractions, the numerators of the fractional roots will differ by the numerators of the fractional expressions for the difference, that is by the common difference, multiplied by the root's denominator, which denominator, in the case of rational roots, is an integral divisor of the highest term.

Amongst the rational and irrational divisors of those quantities, $(128m+64P+32Q+16R+8S+4T+2V+W)$, $(m+P+Q+R+S+T+V+W)$, (W), $(-m+P-Q+R-S+T-F+W)$, &c. &c. there should be as many arithmetical series as is the number of dimensions of the equation; for, so many roots are successively diminished from the nature of the operation. But only the terms of the rational series, and the rational products or contents of the irrationals, can be found among the

the rational divisors, viz. if the roots of the given equation be $\frac{-a}{D}$, $\frac{-b}{D}$, $\frac{-c}{D}$, &c. and A is any substituted term, the corresponding terms in the different series of the increasing numerators of the roots will, with their signs changed, be found among the corresponding terms of the decreasing series of divisors, which terms, by putting D generally for any denominator, will be $A \times D + a$, $A \times D + b$, $A \times D + c$, &c.; and the products or contents of these, will be $A^n \times D^n + A^{n-1} \times D^{n-1}\overline{a+b+c+\&c.} + A^{n-2} \times D^{n-2}\overline{ab+ac+\&c.} + A^{n-3} \times D^{n-3}\overline{abc+abd+\&c.+\&c.} + abcde$ &c.; and here all the powers of the substituted number, descending from the number of dimensions of the polynome required, are connected with the different combinations of the numerators of n number of roots with their signs changed, that is with the numerators of the co-efficients of a polynomial divisor of the proposed equation, their common denominator being the product of the denominators of n roots, since the numerators of such co-efficients are made up of members each of which is a product of the numerators of roots, with their signs changed, multiplied into all the rest of the n denominators but their own.

When the substituted numbers are the terms of an arithmetical series, we shall have their powers multiplied, in the successive results, by the several co-efficients of the divisor required, and if we take the differences of such results, and the

the differences of those differences, &c. as often as the index of the highest power denotes, the co-efficients which are connected with the inferior powers will be exterminated, and the highest co-efficients alone will be involved in the last differences, with the constant differences of the powers. For, all the n^{th} differences of the n^{th} powers of numbers in arithmetical progression are constant and $= 1 \times 2 \times 3 \times 4 \ldots (n-1).n \times d^n$

A general statement of Newton's rules for finding divisors can be easily deduced from the foregoing observations, as follows:

Substitute successively for x in the proposed, the terms of an arithmetical series, 3, 2, 1, 0, -1, -2, -3, until the number of terms is greater then the index of the divisor required; place the numbers resulting from the substitution with all their divisors, as well affirmative as negative, opposite to the correspondent terms of the substituted series; take the differences of those divisors, and the differences of their differences, &c.; if the differences of any series of divisors be common, when the number of orders of differences taken is equal to n or the index of the polynome sought, that difference being divided by the last difference of the n^{th} powers of the terms of the natural series or by $1.2.3 \ldots n$, the quotient should be a divisor of the highest term of the proposed, and if so should be made the common denominator of the co-efficients of

of the polynomial divisor, or the co-efficient of the highest term of that divisor. Subduct the divisors of the above results from the powers of the correspondent terms of the arithmetical series (n being the index of the powers) multiplied into the last found numeral divisor of the highest term. Thus you will have subducted the divisors from their own first members: hence there will be found only the $n-1^{th}$ powers for the highest powers in the remainders, and the $n-1^{th}$ differences of the remainders will be constant, and being divided by $1.2.3...\overline{n-1}$, the quote is the numerator of the co-efficient of the second term with the signs changed, (for the signs of the co-efficients are changed by the subduction of the divisors from their own first members). Subduct the last found numerator, with the sign changed, multiplied into the correspondent powers of the natural numbers whose index is $n-1$, from the first remainders. If the order of differences of these or of the second remainders be constant when the number of orders is equal to $n-2$, that difference being divided by $1.2.3...\overline{n-2}$, the quote is the numerator of the third co-efficient with the sign changed.

In general, if the numerator of a co-efficient with the sign changed, whose distance from the first term is $m-1$, be derived from the last differences of the $m-1^{th}$ remainders, divided by $1.2.3...(n-\overline{m-1})$ as we have shewn where that distance is

is 1 or 2; in like manner the numerator of a co-efficient with the sign changed, whose distance from the first is m, will be derived from the last differences of the remainders which are deduced from the former remainders by subducting from them the numerator of the co-efficient last found, with the sign changed, multiplied into the powers of the correspondent terms of the natural series, the index of which powers is $n-\overline{m-1}$.

If $1.2.3\ldots n\times D$ is the n^{th} difference of the divisors of the absolute term, or if D is otherwise sought among the numeral divisors of the highest term of the proposed, and if the last differences of the first, second, third, &c. remainders, be viz. $1.2.3\ldots\overline{n-1}\times -p$, $1.2.3\ldots\overline{n-2}\times -q$, $1.2.3\ldots\overline{n-3}\times -r$, &c. The polynomial divisor of n dimensions will be $x^n+\frac{p}{D}x^{n-1}+\frac{q}{D}x^{n-2}+\frac{r}{D}x^{n-3}+\&c.$ or $Dx^n+px^{n-1}+qx^{n-2}+rx^{n-3}$ $+\&c.+abcd\,\&c.=0$

Note. The quantities called remainders are the divisors subducted from the sum of their own members already discovered. The numerator of the required co-efficient must be made to stand in the highest place in those expressions whose differences are to be taken, and this is effected as above by taking away the higher powers which were connected with the preceding co-efficients.

An abridged mode of finding divisors may be more simply deduced from the general expression, in the same manner as Gravesand has done in his example for the cubical divisor. For, the following being the form of the divisor, viz. $A^n \times D^n + A^{n-1} \times D^{n-1}\overline{a+b+c+\&c.} + A^{n-2} \times D^{n-2} \times \overline{ab+ac+\&c.}$ $+ \&c... + A \times D\overline{abc\,\&c. + acd\,\&c. + \&c.} + abcde$ &c. If we therefore begin the operation by subducting the sum of the first and last members, or $A^n \times D^n + abcde$ &c. ($A^n \times D^n$ being found as before, and *abcde* &c. being found opposite to cypher) and, if we then divide the remainder by A, we shall thus have depressed the indices of the powers of the substituted numbers by 2, and therefore the differences to be taken will be fewer. Hence, in the case of a divisor of three dimensions, or a divisor of four dimensions, whose second term is wanting, the quotes or depressed remainders are in arithmetical series, and thus, two co-efficients are discovered together, the second co-efficient being the common difference of the arithmetical series, and the penultimate being the term of that series which should correspond to cypher. This term, however, is not immediately discoverable opposite to cypher, and although in the cubical divisor it is always to be known, being the basis of the arithmetical series, yet the term is not really expressed, (for the divisor *abcde* &c. being subducted from itself would leave cypher, and as this corresponds to the substitution of cypher, the division of the remainder by the substituted

substituted quantity, or of cypher by cypher, will give any finite quotient, and the number being indeterminate is understood till it is discovered by continuing the law of the series.) But in finding divisors of higher dimensions where those depressed remainders or quotes are not in arithmetical series, and where, in order to obtain arithmetical series, the differences of the quotes are to be taken, no such quotes can be understood, and therefore the substitution cannot then be continued beyond cypher. But if we substitute the terms of the natural series, from the index of the polynome to unit, when the arithmetical series shall be obtained from the higher order of differences of the quotes or depressed remainders, the term opposite to cypher is found by continuing the law of the differences, and thence the law of the series for one step farther. From those principles, as in the following rule, we discover two co-efficients of the divisor at once.

Substitute for the unknown letter in the proposed the terms of the natural series, descending from the index of the divisor required to unity; adding the divisors opposite to cypher to the divisors of the highest term, multiplied into the n^{th} powers of the natural numbers, let the sums be respectively taken from the divisors of the results of substitution of such corresponding natural numbers; divide the remainders by those corresponding substituted numbers, and if the difference of the quotes be common, when the number of orders is $n-2$, the common difference divided by $1.2.3...\overline{n-2}$ is the numeral o-efficient of the second term of the divisor of n dimensions;

 let

let k be the extreme quote corresponding to 1, and $-k_1$, $-k_2$, $-k_3$, &c. the extreme differences of those quotes of the first, second, third, &c. orders, by continuing the differences beginning with the penultimate differences, we shall continue the terms of the series, and the term opposite to cypher will be found $k+k_1+k_2+k_3+$ &c.; and because the member which alone is not multiplied by the substituted number in the quotes, is the co-efficient of the penultimate term of the divisor, that co-efficient should be the term found opposite to cypher.

In general, the co-efficient whose distance from the first is m, along with the co-efficient whose distance from the last is also $=m$, will be derived, in like manner as above, from the quotes which result from dividing by the corresponding substituted numbers, the remainders, after the quotes in the preceding step are diminished by the co-efficient whose distance from the last is m, and by the product of the co-efficient whose distance from the first is the same, multiplied into the powers of the correspondent natural numbers, whose index is $n-2m$.

If the last differences of the quotes in the first, second, &c. steps, be $1.2.3...\overline{n-2}p$, $1.2.3...\overline{n-4}q$, &c.; and if, in the series of the first, second, &c. quotes or depressed remainders, the terms opposite to cypher be $k+k_1+k_2+$ &c. $l+l_1+l_2+$ &c. &c. the polynomial divisor is $Dx^n+px^{n-1}+qx^{n-2}+\text{\&c.}+(l+l_1+\text{\&c.})x^2+(k+k_1+\text{\&c.})x+abcd$ &c.

CHAP.

CHAP. II.

The practical Application of the foregoing general Rules.

FOR an example of the method of finding divisors, let the proposed equation be $18x^7-15x^6-12x^5+10x^4-108x^3+90x^2+42x-35=0$, and let the roots be diminished by the terms of the natural series:

	Results of substitution or absolute terms:	Some of the rational divisors:						Decreasing arithmetical series.
3	24310	13,	34,	442,	55,	715,	1870,	13, $3\sqrt{3}\pm\sqrt{-7}$, &c.
2	665	7,	19,	133,	5,	35,	95,	7, $2\sqrt{3}\pm\sqrt{-7}$, &c.
1	-10	1,	10,	10,	-1,	-1,	-10,	1, $\sqrt{3}\pm\sqrt{-7}$, &c.
0	-35	-5,	7,	-35,	1,	-5,	+7,	-5, $\pm\sqrt{-7}$, &c.
-1	110	-11,	10,	-110,	-1,	11,	-10,	-11, $-\sqrt{3}\pm\sqrt{-7}$, &c.
-2	-1615	-17,	19,	-323,	5,			
-3	-43010							

Thus, to find a divisor of one dimension:

$$
\begin{array}{rcccl}
13 & & & 3\times6-13=5 & \\
 & 6 & & & \\
7 & & & 2\times6-\ 7=5 & \\
 & 6 & & & \\
1 & & D=6 & 1\times6-\ 1=5 & p=-5 \\
 & 6 & & & \\
-5 & & & 0\times6+\ 5=5 & \\
 & 6 & & & \\
-11 & & & -1\times6+11=5 &
\end{array}
$$

The divisor is $x-\frac{5}{6}$ or which is the same thing, $6x-5=0$.

To

To find a divisor of two dimensions:

$$\begin{array}{rrr} 34 & & \\ & 15 & \\ 19 & & 6 \\ & 9 & \\ 10 & & 6 \\ & 3 & \\ 7 & & 6 \\ & -3 & \\ 10 & & \end{array} \qquad D=\frac{6}{1.2}=3 \qquad \begin{array}{l} 9\times 3-34=-7 \\ 4\times 3-19=-7 \\ 1\times 3-10=-7 \\ 0\times 3-\ 7=-7 \\ 1\times 3-10=-7 \end{array} \quad \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \quad p=0 \quad \begin{array}{l} -7-3\times 0=-7 \\ -7-2\times 0=-7 \\ -7-1\times 0=-7 \end{array} \quad q=7$$

The divisor is $x^2+\frac{7}{3}$ or $3x^2+7=0$.

To find a divisor of three dimensions:

$$\begin{array}{rrrr} 442 & & & \\ & 309 & & \\ 133 & & 186 & \\ & 123 & & 108 \\ 10 & & 78 & \\ & 45 & & 108 \\ -35 & & -30 & \\ & 75 & & \\ -110 & & & \end{array} \qquad \begin{array}{l} 27\times 18-442=44 \\ 8\times 18-133=11 \\ 1\times 18-\ 10=\ 8 \\ 0\times 18+\ 35=35 \\ -1\times 18+113=92 \end{array} \quad \begin{array}{rr} 33 & \\ & 30 \\ 3 & \\ & 30 \\ -27 & \\ & 30 \\ -57 & \end{array} \qquad \begin{array}{l} 44-9\times 15=-91 \\ 11-4\times 15=-49 \\ 8-1\times 15=-7 \\ 35-0\times 15=\ 35 \\ 92-7\times 15=\ 77 \end{array} \quad \begin{array}{r} -42 \\ -42 \\ -42 \\ -42 \end{array}$$

$$D=\frac{108}{1.2.3}=18 \qquad p=\frac{-30}{1.2}=-15 \quad q=42,\ r=-35$$

The divisor is $x^3-\frac{15}{18}x^2+\frac{42}{18}x-\frac{35}{18}$ or $18x^3-15x^2+42x-35=0$

To find a divisor of four dimensions, according to the general statement of Gravesand's method, or the abridged mode which was given above:

$$\begin{array}{rrrrr} 55 & & & & \\ & 50 & & & \\ 5 & & 44 & & \\ & 6 & & 36 & \\ -1 & & 8 & & 24 \\ & -2 & & 12 & \\ 1 & & -4 & & 24 \\ & 2 & & -12 & \\ -1 & & 8 & & \\ & -6 & & & \\ 5 & & & & \end{array} \qquad \begin{array}{l} 55-(81+1)=-27 \\ 5-(16+1)=-12 \\ -1-\ (1+1)=-3 \\ 1-\ (0+1)=\ 0 \end{array} \ \text{Divide by substituted numbers.} \ \begin{array}{rr} -9 & \\ & -3 \\ -6 & \\ & -3 \\ -3 & \\ & -3 \\ 0 & \end{array} \ \begin{array}{c} 0 \\ 0 \end{array} \quad \begin{array}{l} -9-0=-9 \\ -6-0=-6 \\ -3-0=-3 \end{array} \ \text{Divide by substituted numbers.} \ \begin{array}{r} -3 \\ -3 \\ -3 \end{array}$$

$$p=0 \qquad r=0 \qquad q=-3$$

$$D=\frac{24}{1\ 2\ 3.4}=1$$

The divisor is $x^4-3x^2+1=0$

From

In like manner, to find a divisor of six dimensions, if the substituted terms were begun with 6, 5, 4, 3, 2, &c. the corresponding results would give the series of divisors, 136745, 45182, 11495, 1870, 95, &c.

		Divide by substituted terms the 4th differences of the quotes. $=0$.					
$136745-(3\times 6^6+7)=$	-3240		-540				
				-200			
$45182-(3\times 5^6+7)=$	-1700		-340		-60		
				-140		-12	
$11495-(3\times 4^6+7)=$	-800		-200		-48		0
				-92		-12	
$1870-(3\times 3^6+7)=$	-320		-108		-36		0
				-56		-12	
$95-(3\times 2^6+7)=$	-104		-52		-34		
				-22			

$$0+12-36+56-52=-20$$

$k+k_1+k_2+k_3+k_4=0+12-34+22-20=0$ and $p=0$

		Divide by substituted terms:			
$-540-(0+0)=$	-540		-90		
				-22	
$-340-(0+0)=$	-340		-68		-4
				-18	
$-200-(0+0)=$	-200		-50		-4
				-14	
$-108-(0+0)=$	-108		-36		-4
				-10	
$-52-(0+0)=$	-52		-26		-4
				-6	
			-20		

$l+l_1+l_2=-4+6-20=-18$, $q=\frac{-4}{1.2}=-2$.

The divisor is $3x^6-2x^4-18x^2+7=0$.

Thus, from 13, 7, 1, —5, —11, we have derived the divisor $6x-5=0$, and from the numbers 34, 19, 10, 7, 10, we have the divisor $3x^2+7=0$. But as these are binomial divisors, and as the usual rules extend only to divisors of one, two, and three dimensions, from them the equation does not appear to admit a trinomial divisor. However, according to the rules for finding a divisor of four dimensions, from the numbers

55,

35, 5, —1, 1, —1, 5, we have a trinomial divisor $x^4-3x^2+1=0$, whose roots can be discovered by resolving a quadratic equation. Also the divisor $18x^3-15x^2+42x-35=0$, which is deduced from the numbers 442, 133, 10, —35, &c. is the only quadrinomial derived by the particular rules. But, by the general rules, from the numbers 1870, 95, —10, 7, —10, 95, we find the quadrinomial $3x^6-2x^4-18x^2+7=0$, which is reducible to a cubic.

A divisor of five dimensions is found from the numbers 715, 35, —1, —5, &c. and although, in this example, from the two first divisors we may investigate the others by help of division, yet the rule discovers them immediately, and in an equation having no divisors of such low dimensions, the usual rules would be inadequate to the discovery of the others.

Sometimes no divisor of a given number of dimensions can be found, which shall succeed in the division, viz. when the content of so many roots is not found among the rational divisors of the absolute term. However, the content of yet a greater number of roots may be rational, and among all the divisors, rational and irrational, there will be always as many arithmetical series as there are roots, for all the roots are necessarily diminished, as we have shewn, from the nature of transformation.

When some of the divisors of the absolute terms of the given equation are odd, and others even, we can, *prima facie*, reduce

reduce to narrower limits the divisors of the other absolute terms which stand opposite to the even substituted numbers; let the even divisors alone be compared with the even divisors of the absolute term of the given equation, and the odd alone with the odd. For, *abcde* &c. or the divisor of the absolute term of the given equation, being subducted from $A^n \times D^n + \&c. + A \times D\overline{abc\ \&c. + bcd\ \&c. + \&c.} + abcde\ \&c.$ or from the divisors of the absolute term corresponding to A, the remainder should be divisible by A. Hence, that the even remainders may correspond to even substituted terms, the divisors to be compared should be together even, or together odd; for, the sum or difference of an odd and an even number cannot be even, and therefore cannot be divisible by an even number.

Thus, in Newton's example, (which shall be given immediately) 14, which is opposite to cypher, cannot be compared with 19, which stands opposite to 2; nor 7, opposed to cypher, with —38, opposed to 2.

From the above general statement, we are enabled, *a priori*, (as soon as we shall have obtained those numeral divisors whose differences afford arithmetical series) to try, without the trouble of division, whether polynomes can from thence be deduced, which shall really divide. For, if an equation can be divided by any compound divisor, it will also be divisible by the quotient, or by the polynome whose index is the difference of the indices of the equation, and of the compound

divisor. And, in a similar manner as by the general rule we deduced a divisor from the series of contents of the numerators of some roots, the signs of those roots being changed, so should the quote be discoverable from the series of contents of the remaining roots, with their signs changed. Therefore, when we have found a series of numeral divisors which have a constant order of differences, if such coincidence with the rule be not casual, the numeral co-factors of those divisors should also coincide with the rule for finding a polynomial divisor whose index is the complement of that of the index of the polynome sought, to that of the given equation. The nighest co-efficient of the new polynome should be the highest co-efficient of the given equation divided by the highest co-efficient of the polynome required, and the co-efficients of the second terms of both polynomes should constitute a sum equal to the co-efficient of the second term of the equation.

Thus, in Newton's example, $3y^5-6y^4+y^3-8y^2-14y+14$, substitute for y the terms of the arithmetical series:

	Results of Substitution:	Numeral divisors:				Take third differences of the divisors:					Take third differences of the divisors:			
3	170	34,	10,	17,	5,	5	7	8	6		17	55	98	132
2	−38	19,	1,	−38,	−2,	−2	−1	2	6	$\frac{6}{1.2.3}=1$	−38	−43	−34	−12
1	−10	10,	−2,	5,	−1,	−1	−3	−4			5	−9	−22	
0	14	7,	1,	14,	2,	2	1				14	13		
−1	10	10,	10,	1,	1,	1					1			

Here the divisors 34, 19, 10, 7, 10, multiplied by the corresponding divisor 5, —2, —1, 2, 1, give the respective absolute terms 170, —38, —10, 14, 10. Now, since the divisors

divisors 34, 19, 10, 7, 10, give us $3y^2+7$, if such is a true divisor, the co-factors 5, —2, —1, 2, 1, should supply (as they do) a trinomial whose highest co-efficient is unit.

But although the second series 10, 1, -2, 1, 10, would apparently give the divisor $3y^2—6y+1$, yet the co-factors 17, —38, 5, 14, 1, will give us no cubical divisor, those co-factors not having the third differences common: therefore we reject the series 10, 1, —3, 1, 10, as casual, and not arising from the nature of the operation.

To reject those numbers 10, 1, —2, 1, 10, Newton continues that series, according to the required law, by continuing the terms of the arithmetical series of the remainders. Now, of the series 10, 1, —2, 1, 10, 25, the new term 25, is found not to divide —190, which results from the substitution of —2. But the method which is above given detects the fortuitous divisors, without the trouble of continuing them, which is a particular advantage, if the fortuitous series should not break off for the several terms. The co-factors are immediately pointed out by the numeral divisors to be tried, and besides the greater facility of numerical subduction than of the division of algebraical quantities, it is also advantageous that the above can be tried when the numerical divisors are first discovered to have a constant order of differences, but algebraical division can only be tried when all the co-efficients of the polynome are completed.

 In

In substituting the terms of the natural series, one of the terms being cypher, the process of involution is easier, not only because of the smallness of the numbers, but also the same process of involution will serve for the substitution of affirmatives and negatives. For, in general, in an arithmetical series, when the next terms to cypher, or the two lowest terms, are the same, but with contrary signs, all the terms below cypher are the same with those above it, excepting the difference of sign. If an affirmative quantity be substituted for x, and if we make one sum of the terms in which the indices are even, and another sum of the terms in which the indices are odd, both sums, added together, will be the result of the substitution of the affirmative, and the sum of the terms containing even powers, less by the sum of the terms containing the odd powers, will be the result of the substitution of the same quantity, with a negative sign.

Yet we would not always substitute the same number of affirmative and negative terms. For, if all the terms of the given equation are affirmative, we shall have lesser results by substituting negatives for x: for the roots being all negative, if we diminish them by negative quantities, we shall bring them nearer to cypher. And if the terms are alternately affirmative and negative, the roots being all affirmative, we should diminish them by affirmative quantities. If the roots of the equation are great, and the absolute term thereof considerable, by considerably diminishing those roots, that is, by

by substitution of large numbers, we may diminish the results; but if the absolute term is not great, the smaller the numbers substituted, the less the results: and this, with the facility of involving small numbers, would make us to prefer the substitution of the natural numbers near cypher.

When the roots are integral, by substituting the terms of the natural series, (which contains all integral numbers), we may be able, by mere substitution, to discover those roots: for, when the root is substituted for the unknown, the result is equal to cypher, since the absolute term of the transformed is equal to cypher when the root is diminished by a quantity equal to itself. When there are fractional roots, we might diminish the roots by an arithmetical series of fractions, and find results equal to cypher, but the substitution of these, and the ridding the terms of denominators, would be the same as to multiply the roots, and then substitute the natural numbers. Thus also, when all the roots have a common factor, we might substitute multiples of natural numbers, or use the natural numbers multiplied by that common factor; but since the results of substitution or the absolute terms, which are the products of the roots with their signs changed, would, in the transformed, be all divisible by the n^{th} power of the common factor, after this division, the result would be the same as if we had first divided the roots by their common factor, and substituted the natural numbers themselves.

The

The particular method of Newton will be easily understood, after the general rules which have been delivered above. I shall conclude this Essay with transcribing the rules for finding divisors, which he has given in his " Universal Arithmetic." His method, in the first of the following passages, will be applicable, as we have seen, to the rational divisors of one dimension; that in the second, will extend to the rational quadratic divisors, or divisors of two dimensions; the third of those passages gives us a general view of the method of finding divisors of n dimensions, the demonstration of which has been given in the former Chapter of this Essay; the concluding paragraph will appear from the observations which I have made in the foregoing page. I shall state all those passages in the original Latin, which is clearer, perhaps, than any English translation could be, on the subject.

Si quantitas postquam divisa est per omnes simplices divisores manet composita, & suspicio est eam compositum aliquem divisorem habere, dispone eam secundum dimensiones literæ alicujus quæ in eâ est, & pro litera illa substitue sigillatim tres vel plures terminos hujus progressionis arithmeticæ, 3, 2, 1, 0, —1, —2, ac terminos totidem resultantes una cum omnibus eorum divisoribus statue e regione correspondentium terminorum progressionis, positis divisorum signis tam affirmativis quam negativis. Dein e regione etiam statue progressiones arithmeticas quæ per omnium numerorum divisores percurrunt pergentes a majoribus terminis ad minores eodem

eodem ordine quo termini progressionis 3, 2, 1, 0, —1, —2, pergunt, & quarum termini differunt vel unitate vel numero aliquo qui dividit altissimum terminum propositæ quantitatis. Si qua occurrit ejusmodi progressio, iste terminus ejus qui stat e regione termini 0 progressionis primæ, divisus per differentiam terminorum, & cum signo suo annexus literæ præfatæ, componet quantitatem per quam divisio tentanda est.

Si nullus occurit hac methodo divisor, vel nullus qui dividit propositam quantitatem, concludendum erit quantitatem illam non admittere divisorem unius dimensionis. Potest tamen fortasse, si plurium sit quam trium dimensionum, divisorem admittere duarum. Et si ita, divisor ille investigabitur hac methodo. In quantitare illa pro litera substitue, ut ante, quatuor vel plures terminos progressionis hujus 3. 2. 1. 0. —1. —2. —3. Divisores omnes numerorum resultantium sigillatim adde & subduc quadratis correspondentium terminorum progressionis illius ductis in divisorem aliquem numeralem altissimi termini quantitatis propositæ, & summas differentiasque e regione progressionis colloca. Dein progressiones omnes collaterales nota quæ per istas summas differentiasque percurrunt. Sit $\mp$C terminus istiusmodi progressionis qui stat e regione 0 progressionis primæ, $\mp$B differentia quæ oritur subducendo $\mp$C de termino proxime superiori, qui stat e regione termini 1 progressionis primæ, A prædictus termini altistimi divisor numeralis, & l litera quæ in quantitate proposita est, & erit $All \pm Bl \pm C$ divisor tendandus.

Si

Si nullus inveniri potest hoc pacto divisor qui succedit, concludendum est quantitatem propositam non admittere divisorem duarum dimensionum. Posset eadem methodus extendi ad inventionem divisorum dimensionum plurium, quærendo in prædictis summis differentiisque progressiones non arithmeticas quidem sed alias quasdam quarum terminorum differentiæ primæ, secundæ, tertiæ, &c. sunt in arithmetica progressione.

Ubi in quantitate proposita duæ sunt literæ, & omnes ejus termini ad dimensiones æque altas ascendunt; pro una istarum literarum pone unitatem, dein per regulas præcedentes quære divisorem, ac divisoris hujus comple deficientes dimensiones restituendo literam illam pro unitate.

ERRATA.

Page	Line	
16 —	1	at top, for *are displeasing*, read *it is displeasing*.
18 —	2	from bottom, for *non ti timo,* read *non ti temo,* &c.
19 —	1	from bottom, for *Pussignol,* read *L'ussignol.*
45 —	13	from top, for *is found,* read *is formed.*
83 —	16	from top, for *with blood,* &c. read *with the blood,* &c.
155 —	5	from bottom, for *proudence,* read *prudence.*
159 —	8	from top, for *abstactedly,* read *abstractedly.*
184 —	7	from top, for *substraction*, read *substratum.*

188, last line, for *pati durosque labore,* read *et duros perferre labores.*